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# Approximate Models for Wave Propagation Across Thin Periodic Interfaces

Bérangère Delourme[*], Houssem Haddar[†], Patrick Joly[‡]

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**Abstract:** This work deals with the scattering of acoustic waves by a thin ring which contains many regularly-spaced heterogeneties. We provide a complete description of the asymptotic of the solution with respect to the period and thickness of the heterogeneities. Then, we build a simplified model replacing the thin perforated ring by an effective transmission condition. We pay particular attention to the stabilization of the effective transmission condition. Error estimates and numerical simulations are carried out to validate the accuracy of the model.

**Key-words:** asymptotic expansions, effective transmission conditions, homogenization, Helmholtz equation

[*] LETI-CEA Grenoble
[†] INRIA Saclay Defi
[‡] INRIA Rocquencourt POEMS

Centre de recherche INRIA Paris – Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

# Modèles approchés pour la simulation d'ondes à travers des couches minces périodiques

**Résumé :** Ce travail est dédié à l'étude de la propagation d'ondes acoustiques à travers une couche mince circulaire et périodique. La période et l'épaisseur de la couche, proportionnelles à un petit paramètre $\delta$ sontdéveloppement de la solution par rapport au petit paramètre $\delta$. Puis nous en déduisons des modèles approchés dans lesquels la couche périodique est remplacée par une condition de transmission équivalente. Ces modèles, validés théoriquement et numériquement, ont l'avantage d'être mieux adaptés aux simulations numériques que le problème initial.

**Mots-clés :** développements asymptotiques, conditions de transmission équivalentes homogenéisation, équation de Helmholtz

# Introduction

This work is dedicated to the study of asymptotic models associated with acoustic waves scattering from thin rings that contain regularly spaced heterogeneities. We are interested in situations where the thickness of the ring and the distance between two consecutive heterogeneities are very small compared to the wavelength of the incident wave and the diameter of the ring. One easily understands that in those cases, numerical computation of the solution would become prohibitive as the small scale (denoted by $\delta$) goes to 0, since the used mesh needs to accurately follow the geometry of the heterogeneities. In order to overcome this difficulty, we shall derive so-called *Approximate Transmission Conditions*, which are transmission conditions that only involve the traces of the field and of its normal derivative on the boundary of the ring and which yield to approximations of the exact solution that polynomially converge to the exact one (as $\delta \to 0$). The numerical discretization of approximate problems is expected to be much less expensive than the exact one, since the used mesh has no longer to be constrained by the small scale.

The use of such approximate models is a rather classical topic in the modeling of wave propagation phenomena when a (geometrical) small scale is present (a typical reference is [1] from the engineering literature). As explained above, the main idea would be to replace an exact problem, which is difficult and expensive to numerically solve (basically due to the need of local mesh refinement imposed by the small scale), with an approximate one which is numerically much cheaper.

Without being exhaustive, let us indicate some works from the mathematical literature that share similarities with our problem or employed methods. For instance, first order approximate boundary conditions have been derived for electromagnetic scattering problems from perfect conductors coated with periodic thin structures in [2] for the Maxwell equations in planar geometries, in [3] and [4] for the Helmholtz equation in circular geometries. Higher order conditions have been derived in [5] for the Laplace problem and in [6], [7] for the Helmholtz equation. The case of approximate transmission conditions has been studied in [8] and [9] for perforated thin conductors: in [8] the two first terms of the asymptotic expansion of the solution has been obtained. The case of effective transmission conditions modelling highly conductive thin sheets is treated in [10]. The goal of this work is to complement the above mentioned studies in two directions: The first one is to provide a complete description of the asymptotic of the solution with respect to the small parameter (in the case of thin circular periodic interfaces). We shall employ for that purpose the technique of so-called matched asymptotic expansions (see for instance [11], [12] and [13]). The second one is the derivation of *variational* and *stable* approximate interface conditions which are accurate up to $O(\delta^2)$ and up to $O(\delta^3)$ errors. The accuracy of these conditions is theoretically and numerically validated through error analysis and numerical simulations of test problems. The latter also demonstrate the effectiveness of introduced approximate models.

The remaining of this report is organized as follows. In Section 1, we describe the setting of the problem and introduce some notation. Section 2 is dedicated to the construction of a matched asymptotic expansion of the solution. The derivation and error analysis of approximate transmission conditions is done in Section 3. The last section is devoted to numerical simulations and validations of the approximated problems. Finally, in order to facilitate the reading of this report, proofs that are too technical are postponed to appendices.

# 1 Setting of the Problem

In these first investigations, we shall restrict ourselves to the scalar scattering problem modelled by the Helmholtz equation in two dimensions. Let $\mu^\delta$ and $\rho^\delta$ denote the acoustical characteristics of the medium $\Omega$ where $\delta$ is a small parameter that will be introduced later. Then the acoustic field $u^\delta$ satisfies:

$$\nabla \cdot (\mu^\delta \nabla u^\delta) + \omega^2 \rho^\delta u^\delta = f \quad \text{in } \Omega, \tag{1}$$

where $\omega$ denotes the pulsation of time variation and $f$ denotes a given source term.
In order to simplify the exposure of the error analysis we shall assume that $\Omega$ is a disk:

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2, \sqrt{x^2+y^2} < R_e \right\}.$$

To shorten notation, we also introduce the interior domain $\Omega^-$, the exterior domain $\Omega^+$ and the interface $S_{R_0}$ defined by

$$\Omega^- := \left\{ (x,y) \in \mathbb{R}^2, \sqrt{x^2+y^2} < R_0 \right\},$$
$$\Omega^+ := \left\{ (x,y) \in \mathbb{R}^2, R_0 < \sqrt{x^2+y^2} < R_e \right\},$$
$$S_{R_0} := \left\{ (x,y) \in \mathbb{R}^2, \sqrt{x^2+y^2} = R_0 \right\}.$$

Moreover, we replace the radiation condition satisfied by $u^\delta$ by the following impedance condition on the boundary of $\Omega$ denoted by $S_{R_e}$:

$$\frac{\partial u^\delta}{\partial r} + i\omega u^\delta = 0 \text{ on } S_{R_e}. \tag{2}$$

Readers who are familiar with scattering problems can be easily convinced that the theoretical treatment of the scattering problem in $\mathbb{R}^2$ (where (2) is replaced by the Sommerfeld radiation condition) can be deduced with minor modifications.

We assume that the medium $\Omega$ is made of a ring (of mean radius $R_0$) $\Omega_R^\delta := \left\{ |r - R_0| \leq \frac{\delta}{2} \right\}$ plugged into an homogeneous medium ($\rho_\infty$,$\mu_\infty$). This ring contains many regularly spaced heterogeneities in the azimuthal direction ($\theta$) which means in particular that $\mu^\delta$ and $\rho^\delta$ are periodic in the angular variable $\theta$ (see Figure 1).

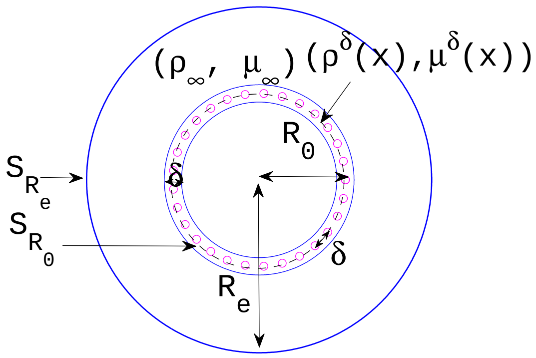



Figure 1: $\Omega$

To clearly define this angular periodicity, we introduce the scaled tangential variable $S$ and the scaled normal variable $\mathcal{V}$ (see Fig. 2):

$$S = \frac{R_0\theta}{\delta} \qquad \text{and} \qquad \mathcal{V} = \frac{r - R_0}{\delta}. \tag{3}$$

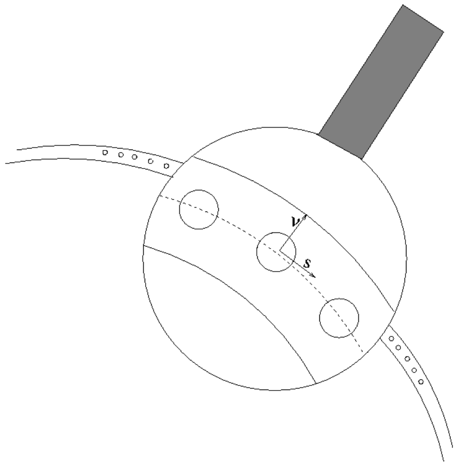



Figure 2: the scaled variables $S$ and $\mathcal{V}$

We assume that there exist two functions $\mu$ and $\rho$ of the scaled variables $S \in \mathbb{R}^+$ and $\mathcal{V} \in \mathbb{R}$, that are independent of $\delta$ and that satisfy:

$$\begin{cases} \mu(\mathcal{V}, S+1) = \mu(\mathcal{V}, S), \\ \rho(\mathcal{V}, S+1) = \rho(\mathcal{V}, S), \end{cases} \quad \text{and} \quad \begin{cases} \mu(\mathcal{V}, S) = \mu_\infty & \text{if } |\mathcal{V}| > \dfrac{1}{2}, \\ \rho(\mathcal{V}, S) = \rho_\infty & \text{if } |\mathcal{V}| > \dfrac{1}{2}, \end{cases} \tag{4}$$

such that

$$\mu^\delta(r,\theta) = \mu(\frac{r-R_0}{\delta}, \theta\frac{R_0}{\delta}) \quad \text{and} \quad \rho^\delta(r,\theta) = \rho(\frac{r-R_0}{\delta}, \theta\frac{R_0}{\delta}) \quad \text{for } r>0 \text{ and } \theta \in [0, 2\pi]. \tag{5}$$

We also make standard assumptions on the bounds for material properties,

$$\begin{cases} \exists\ (\mu_m, \mu_M) \in \mathbb{R}^2, \quad 0 < \mu_m < \mu < \mu_M, \\ \exists\ (\rho_m, \rho_M) \in \mathbb{R}^2, \quad 0 < \rho_m < \rho < \rho_M. \end{cases} \tag{6}$$

Finally we shall assume that the support of the source term $f$ does not intersect the thin ring $\Omega_R^\delta$.

**Philosophy behind approximate interface conditions**

We first recall that the variational problem associated with (1) and (2) can be written as: find $u^\delta \in H^1(\Omega)$ such that

$$a^\delta(u^\delta, v) = L(v) \quad \forall v \in H^1(\Omega), \tag{7}$$

where

$$a^\delta(u,v) = \int_\Omega \left(\mu^\delta \nabla u \cdot \nabla \bar{v} - \omega^2 \rho^\delta u \bar{v}\right)\, dx \;+\; i\omega\mu_\infty \int_{s_{R_e}} u\bar{v}\, ds,$$

$$L(v) = \int_\Omega f\bar{v}\, dx.$$

This problem is well-posed and is stable uniformly with respect to $\delta$:

**Proposition 1.1.** *Problem (7) is well-posed. Moreover, there exists a constant C independent of $\delta$ such that,*

$$\|u^\delta\|_{H^1(\Omega)} \le C \sup_{v \in H^1(\Omega), v \neq 0} \frac{|a^\delta(u^\delta, v)|}{\|v\|_{H^1(\Omega)}} \quad \forall u^\delta \in H^1(\Omega). \tag{8}$$

The proof is standard and is given in section A.1.

We remark that $a^\delta$ can be split into two parts:

$$\bullet \int_{\Omega\setminus\Omega_R^\delta} \left(\mu_\infty \nabla u \cdot \nabla \bar{v} - \omega^2 \rho_\infty u \bar{v}\right)\, dx \;+\; i\omega\mu_\infty \int_{s_{R_e}} u\bar{v}\, ds, \tag{9}$$

$$\bullet \int_{\Omega_R^\delta} \left(\mu^\delta \nabla u \cdot \nabla \bar{v} - \omega^2 \rho^\delta u \bar{v}\right) dx. \tag{10}$$

When approximating problem (7) using finite elements, the main problem comes from the approximation of the term (10), since $\mu^\delta$ and $\rho^\delta$ have fast variations. The goal of approximate transmission conditions would be to replace this term by a boundary integral of the form

$$\int_{S_{R_0}} \mathcal{B}_j^\delta \begin{pmatrix} u(R_0^\pm, \theta) \\ \frac{\partial u}{\partial r}(R_0^\pm, \theta) \end{pmatrix} \cdot \begin{pmatrix} \bar{v}(R_0^\pm, \theta) \\ \frac{\partial \bar{v}}{\partial r}(R_0^\pm, \theta) \end{pmatrix} \, ds, \tag{11}$$

where $\mathcal{B}_j^\delta$ is a local boundary operator that takes into account the characteristics of the medium inside the small ring. Roughly speaking, we say that $\mathcal{B}_j^\delta$ is a transmission operator of order $j$ if (11) approximates (10) up to $O(\delta^{j+1})$ error. The larger is the value of $j$, the more complicated is the expression of $\mathcal{B}_j^\delta$. In theory, one can build $\mathcal{B}_j^\delta$ for any order $j$ but calculations become extremely heavy for $j \geq 3$. We shall restrict ourselves here to $j = 1$ and $j = 2$.

The process of obtaining the $\mathcal{B}_j^\delta$ is based on two main steps.

- In the region $|r - R_0| \gg \delta$ we first prove that the solution has a polynomial asymptotic expansion of the form
$$u^\delta = \sum_{n\in\mathbb{N}} \delta^n u_n.$$
This step is very technical. The rigorous analysis employs the technique of matched asymptotic expansions, and indeed requires introducing the expansion of the field in the region $|r - R_0| \sim \delta$ .

- Then, in order to derive an approximate model of order $j$, we truncate the asymptotic expansion at $n = j$, and consider $u_j^\delta := \sum_{n=0}^{j} \delta^n u_n$. Then we shall observe, from analytical expressions of the near fields, the existence of $\mathcal{B}_j^\delta$ such that

$$\int_{r=R_0} \mathcal{B}_j^\delta \begin{pmatrix} u_j^\delta(R_0^\pm, \theta) \\ \dfrac{\partial u_j^\delta}{\partial r}(R_0^\pm, \theta) \end{pmatrix} \cdot \begin{pmatrix} \bar{v}(R_0^\pm, \theta) \\ \dfrac{\partial \bar{v}}{\partial r}(R_0^\pm, \theta) \end{pmatrix} \, ds = \int_{\Omega_R^\delta} \left(\mu^\delta \nabla u_j^\delta \cdot \nabla \bar{v} - \omega^2 \rho^\delta u_j^\delta \bar{v}\right)\, dx \;+\; O(\delta^{j+1}).$$

As we shall notice, the construction of $\mathcal{B}_j^\delta$ is not unique. The main difficulty is to derive expressions of $\mathcal{B}_j^\delta$ that have "good"stability properties (namely for which the approximate solution satisfies an uniform stability estimate similar to (8)) . Unfortunately we do not have a systematic procedure to derive a "good"expression for $\mathcal{B}_j^\delta$. The final expressions we will provide are motivated by the difficulties encountered when studying the well-posedness of the approximate model associated with the "natural"expressions provided by the asymptotic expansion.

We end this part by introducing some short notation: let $u \in H^1(\Omega^+) \cap H^1(\Omega^-)$. We abbreviate the exterior and interior values of $u$ on the interface $S_{R_0}$ by $u^+$ and $u^-$:

$$u^+(\theta) := u(R_0^+, \theta), \qquad u^-(\theta) := u(R_0^-, \theta).$$

The jump and mean values across $S_{R_0}$ respectively denoted by $[u]$ and $\langle u \rangle$ are defined by:

$$[u] := u^+ - u^-, \qquad \langle u \rangle := \frac{1}{2}\left(u^+ + u^-\right). \tag{12}$$

# 2 Asymptotic Expansion

In order to develop accurate approximate models, we look for a complete description of the asymptotic behavior of the solution when $\delta$ tends to zero: the natural idea is to build an expansion of $u^\delta$ in powers of $\delta$.

The basic method to obtain asymptotic expansions is divided into three main steps: starting from an ansatz we formally derive the formal expansion. In a second step we prove that the terms of the asymptotic expansion are well defined. The last step is to validate the asymptotic expansion by establishing error estimates.

From a technical point of view, we use the method of matched asymptotic expansion. This method has been developed by Van Dick ([14]) to treat singular perturbation problems which arise in fluid mechanics. A standard work on the matched asymptotic expansions applied to the Helmholtz equation can be found in [12],[13] and complex situations are studied in [15]. For recent applications, we refer the reader to [11],[16].

## 2.1 Main Ideas Behind Matched Asymptotic Expansions

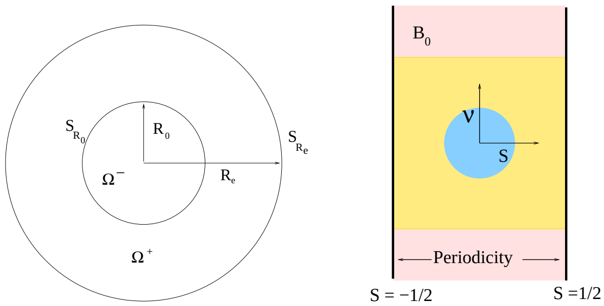
Due to the fast variations with respect to angular coordinates, it is not possible to write an uniform expansion of the solution in the whole domain $\Omega$. Roughly speaking, the solution $u^\delta$ oscillates more rapidly in the vicinity of the periodic ring than far from it. The technique of matched asymptotic expansion consists in separating these two distinct behaviors by expanding first the solution (separately) in the far field zone ( $\dfrac{|r-R_0|}{\delta} \gg 1$) and in the near-field zone ($|r-R_0| \sim \delta$). Then match the two expansions in an intermediate zone $\delta \ll \|r-R_0\| \ll 1$.

### 2.1.1 Ansatz

- Far from the periodic ring, we assume that a standard polynomial expansion holds:

$$u^\delta(r,\theta) = \begin{cases} \displaystyle\sum_{n\in\mathbb{N}} \delta^n u_n^+(r,\theta) & r \gg R_0, \\ \displaystyle\sum_{n\in\mathbb{N}} \delta^n u_n^-(r,\theta) & r \ll R_0. \end{cases} \tag{13}$$

The far field terms $u_n^\pm$ are assumed to be independent of $\delta$. Note also that we assume that $u_n^\pm$ are defined in $\Omega^\pm$ (see figure3(a)).




(a) $\Omega^+$ and $\Omega^-$ (b) $B_0$

Figure 3: Domains for far and near field terms

- Near the periodic ring, we have to take into account the periodicity of $\mu$ and $\rho$. That is why the expansion of $u^\delta$ is more complicated:

$$u^\delta(r,\theta) = \sum_{n\in\mathbb{N}} \delta^n \; U_n(\mathcal{V}, S, \theta) \quad \text{with} \quad \mathcal{V} = \frac{r - R_0}{\delta} \text{ and } S = R_0 \frac{\theta}{\delta}, \tag{14}$$

where the near field terms $U_n$ are defined in $B_0 \times [0, 2\pi]$ where $B_0$ is the periodicity cell (see Figure 3(b)):

$$B_0 := \left\{ (S, \mathcal{V}) \in \mathbb{R}^2, -\frac{1}{2} < S < \frac{1}{2} \right\}. \tag{15}$$

In order to take into account the fast oscillations of $u^\delta$ in the vicinity of the ring, we impose on $U_n$ to be 1-periodic in the tangential variable S. This kind of ansatz is classical in the homogenization theory (see for example [17],[2] [5] for more details).

- The two expansions (14) and (13) are assumed to be also valid in two overlapping zones $\Omega^+_{M,\delta}$ and $\Omega^-_{M,\delta}$ defined by (see Figure 4):

$$\Omega^+_{M,\delta} := \left\{ (r,\theta) \in \mathbb{R}^2, \eta^-(\delta) \le r - R_0 \le \eta^+(\delta) \right\},$$
$$\Omega^-_{M,\delta} := \left\{ (r,\theta) \in \mathbb{R}^2, \eta^-(\delta) \le -(r - R_0) \le \eta^+(\delta) \right\},$$

where, $\eta^\pm$ are chosen such that $0 < \eta^- < \eta^+$ and,

$$\lim_{\delta\to 0} \eta^\pm = 0, \qquad \lim_{\delta\to 0} \frac{\eta^\pm(\delta)}{\delta} = \pm\infty \, . \tag{16}$$

For instance, $\eta^-(\delta) = \sqrt{\delta}$ and $\eta^+(\delta) = 2\sqrt{\delta}$ would be convenient.

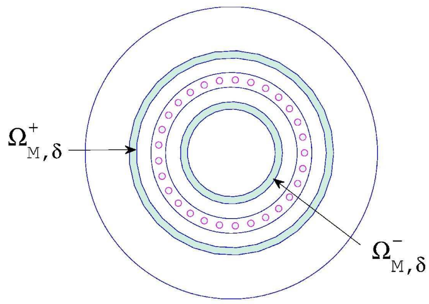



Figure 4: Overlapping zones

Let us notice that, using properties (16) of $\eta$, for the near field, overlapping areas correspond to $\mathcal{V}$ going to $\pm\infty$. On the contrary, for the far field, the overlapping areas correspond to $r \approx R_0$.
A detailed analysis of the behaviour of far and near fields will allow us to find matching conditions that enable far and near field expansions to match in the overlapping zones.

Expansions (13) and (14) will be justified by the error analysis in Section 2.4.

In the two following sections, we shall formally derive the equations satisfied by far fields terms $u_n^\pm$ and near fields terms $U_n$.

### 2.1.2 Far Fields Equations

Substituting $u^\delta$ by its far field expansion (13) in the Helmholtz equation (1) and in the impedance condition (2) and formally separating the different powers of $\delta$, we obtain the equations satisfied by the far fields terms $u_n^\pm$.

$$\boxed{\begin{array}{l} \mu_\infty \triangle u_n^\pm + \omega^2 \rho_\infty u_n^\pm = \left\{ \begin{array}{ll} f & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{array} \right. \quad \text{in } \Omega^\pm, \\ \dfrac{\partial u_n^+}{\partial r} + i\omega u_n^+ = 0 \quad \text{on } S_{R_e}. \end{array}} \tag{17}$$

We emphasize that $u_n^\pm$ are not entirely defined since we did not prescribe yet boundary conditions on $S_{R_0}$: we now have to find transmission conditions between $u_n^+$ and $u_n^-$ on the interface $S_{R_0}$. More precisely, we want to determine the jump $[u_n]$ and the jump of the normal derivative $[\dfrac{\partial u_n}{\partial r}]$ across $S_{R_0}$. These conditions will be obtained from the matching conditions between far and near fields (see section 2.1.4).

### 2.1.3 Near Fields Equations

Let us first remark that $\forall n \in \mathbb{N}$, since $\mathcal{V} = \dfrac{r - R_0}{\delta}$ and $S = \dfrac{R_0 \theta}{\delta}$,

$$\frac{dU_n}{dr} = \frac{1}{\delta} \frac{\partial U_n}{\partial \mathcal{V}},$$

$$\frac{dU_n}{d\theta} = \frac{\partial U_n}{\partial \theta} + \frac{R_0}{\delta} \frac{\partial U_n}{\partial S}.$$

Consequently,

$$\begin{aligned} r^2 \left( \nabla.(\mu^\delta \nabla U_n) + \rho^\delta \omega^2 U_n \right) &= \frac{1}{\delta^2} \left( r^2 \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial U_n}{\partial \mathcal{V}} \right) + R_0^2 \frac{\partial}{\partial S} \left( \mu \frac{\partial U_n}{\partial S} \right) \right) \\ &+ \frac{1}{\delta} \left( r\mu \frac{\partial U_n}{\partial \mathcal{V}} + R_0 \frac{\partial}{\partial S} \left( \mu \frac{\partial U_n}{\partial \theta} \right) + R_0 \mu \frac{\partial^2 U_n}{\partial S \partial \theta} \right) \\ &+ \mu \frac{\partial^2 U_n}{\partial \theta^2} + r^2 \rho \omega^2 U_n. \end{aligned}$$

Replacing $r$ by $R_0 + \delta \mathcal{V}$ gives

$$\begin{aligned} r^2 \left( \nabla.(\mu^\delta \nabla U_n) + \rho^\delta \omega^2 U_n \right) &= \frac{1}{\delta^2} \left( R_0^2 \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial U_n}{\partial \mathcal{V}} \right) + R_0^2 \frac{\partial}{\partial S} \left( \mu \frac{\partial U_n}{\partial S} \right) \right) \\ &+ \frac{1}{\delta} \left( 2R_0 \mathcal{V} \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial U_n}{\partial \mathcal{V}} \right) + R_0 \mu \frac{\partial U_n}{\partial \mathcal{V}} + R_0 \frac{\partial}{\partial S} \left( \mu \frac{\partial U_n}{\partial \theta} \right) + R_0 \mu \frac{\partial^2 U_n}{\partial S \partial \theta} \right) \\ &+ \left( \mathcal{V} \mu \frac{\partial U_n}{\partial \mathcal{V}} + \mathcal{V}^2 \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial U_n}{\partial \mathcal{V}} \right) + \mu \frac{\partial^2 U_n}{\partial \theta^2} + R_0^2 \rho \omega^2 U_n \right) \\ &+ \delta \left( 2\mathcal{V} R_0 \rho \omega^2 U_n \right) \\ &+ \delta^2 \left( \mathcal{V}^2 \rho \omega^2 U_n \right). \end{aligned} \tag{18}$$

Since $u^\delta$ is solution of the homogeneous Helmholtz equation in the vicinity of the periodic ring

$$\sum_{n\in\mathbb{N}} \delta^n r^2 \left(\nabla.(\mu^\delta \nabla U_n) + \rho^\delta \omega^2 U_n\right) = 0.$$

Introducing (18) in the previous equation, and collecting terms of $\delta^n$, we obtain equations for the near fields $U_n$ that can be written in the following form (we adopt the convention that $U_n = 0$ if $n$ is negative):

$$\boxed{\mathcal{A}_0 U_n = -\mathcal{A}_1 U_{n-1} - \mathcal{A}_2 U_{n-2} - \mathcal{A}_3 U_{n-3} - \mathcal{A}_4 U_{n-4} \quad \text{in } B_0} \tag{19}$$

where

$$\mathcal{A}_0 U := R_0^2 \left( \frac{\partial}{\partial \mathcal{V}}(\mu \frac{\partial U}{\partial \mathcal{V}}) + \frac{\partial}{\partial S}(\mu \frac{\partial U}{\partial S}) \right),$$

$$\mathcal{A}_1 U := \mathcal{A}_1^\theta (\frac{\partial U}{\partial \theta}) + \mathcal{A}_1^0 U,$$

$$\text{with} \qquad \mathcal{A}_1^\theta U := R_0 \frac{\partial \mu U}{\partial S} + \mu R_0 \frac{\partial U}{\partial S},$$

$$\mathcal{A}_1^0 U := \frac{1}{\mathcal{V}} \left( 2R_0 \mathcal{V}^2 \frac{\partial}{\partial \mathcal{V}}(\mu \frac{\partial U}{\partial \mathcal{V}}) + R_0 \mathcal{V} \mu \frac{\partial U}{\partial \mathcal{V}} \right),$$

$$\mathcal{A}_2 U := \mathcal{A}_2^{\theta\theta} (\frac{\partial^2 U}{\partial \theta^2}) + \mathcal{A}_2^0 (U),$$

$$\text{with} \qquad \mathcal{A}_2^{\theta\theta} U := \mu U,$$

$$\mathcal{A}_2^0 U := \mathcal{V}^2 \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial U}{\partial \mathcal{V}} \right) + \mu \mathcal{V} \frac{\partial U}{\partial \mathcal{V}} + \omega^2 \rho R_0^2 U,$$

$$\mathcal{A}_3 U := \mathcal{V}(2R_0 \omega^2 \rho U),$$

$$\mathcal{A}_4 U := \mathcal{V}^2 (\omega^2 \rho U).$$

To entirely define the near fields, we need to prescribe their behaviour for large $\mathcal{V}$. The matching conditions allow us to determine these behaviours. A modal expansion of the near field terms in the overlapping zones (i.e for large $\mathcal{V}$) makes possible the writing of these matching conditions.

**Modal Expansion of $U_n$ for $|\mathcal{V}| > \frac{1}{2}$**

The following proposition establishes the behaviour of $U_n$ for large $\mathcal{V}$. An important point to notice here is the fact that the impact of the periodic ring is localized in its vicinity.

**Proposition 2.1.** *Let $U_n$ be a function in $C^\infty(]0, 2\pi[) \times \mathcal{C}^\infty \left( \left\{ (\mathcal{V}, S) \in \mathbb{R}^2 \text{ such that } |\mathcal{V}| > \frac{1}{2} \right\} \right)$ which satisfies (19), which is 1-periodic in $S$ and which is non-exponentially increasing for large $\mathcal{V}$. Then, the* *behaviour of $U_n$ for large $\mathcal{V}$ is given by*

$$U_n(\mathcal{V},S,\theta)=\sum_{k=0}^{n+1}C^+_{n,k}(\theta)\;\mathcal{V}^k\;+\sum_{l\in\mathbb{Z},l\neq 0}(\sum_{k=0}^{n}B^+_{n,l,k}(\theta)\mathcal{V}^k)\;e^{-2\pi|l|\mathcal{V}}\;e^{2i\pi lS}\qquad \textit{for } \mathcal{V}>\frac{1}{2},\quad (20)$$

$$U_n(\mathcal{V},S,\theta)=\sum_{k=0}^{n+1}C^-_{n,k}(\theta)\;\mathcal{V}^k\;+\sum_{l\in\mathbb{Z},l\neq 0}(\sum_{k=0}^{n}B^-_{n,l,k}(\theta)\mathcal{V}^k)\;e^{2\pi|l|\mathcal{V}}\;e^{2i\pi lS}\qquad \textit{for } \mathcal{V}<-\frac{1}{2},\quad (21)$$

*where $B^{\pm}_{n,l,k}(\theta)$ denote some constants that only depend on $\theta$.*

The proof is done by induction. The main idea is to write $U_n$ as Fourier series:

$$U_n(\mathcal{V},\theta,S):=\sum_{l\in\mathbb{Z}}(U_n)_l(\mathcal{V},\theta)\;e^{2i\pi lS}.$$

Then solve the equation for $(U_n)_l$ where $\theta$ is a parameter. The complete proof is given in Appendix A.2.

**Notation 2.2.** *In order to shorten notation, we shall denote by $o(\mathcal{V}^{-\infty})$ any term that can be written in the form:*

$$\sum_{l\in\mathbb{Z},l\neq 0}(\sum_{k=0}^{n}B^+_{l,k}(\theta)\mathcal{V}^k)\;e^{-2\pi|l||\mathcal{V}|}\;e^{2i\pi lS}.$$

*where $B_{l,k}$ does not depend on $S$ and $\mathcal{V}$.*
*The expression of $o(\mathcal{V}^{-\infty})$ may vary from one line to another but $o(\mathcal{V}^{-\infty})$ always satisfies*

$$\lim_{\mathcal{V}\to\pm\infty}\mathcal{V}^m\times o(\mathcal{V}^{-\infty})=0\quad\forall\,m\in\mathbb{R}.$$

*Moreover, $o(\mathcal{V}^{-\infty})$ is periodic with respect to $S$ and $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}o(\mathcal{V}^{-\infty})\;dS=0.$*

For instance we can write,

$$U_n(\mathcal{V},S,\theta)=\sum_{k=0}^{n+1}C^{\pm}_{n,k}(\theta)\;\mathcal{V}^k+o(\mathcal{V}^{-\infty})\quad\text{for }\pm\mathcal{V}>\frac{1}{2}.$$

### 2.1.4 Matching Conditions Between Far-fields and Near-fields

We are now in a position to derive the matching conditions. We consider the overlapping zone in $\mathbb{R}^2$

$$\Omega_{M,\delta}:=\left\{(r,\theta)\in\mathbb{R}\times[0,2\pi],\eta^-(\delta)\leq|r-R_0|\leq\eta^+(\delta)\right\}.$$

where the functions $\eta^-$ and $\eta^+$ verify (16) (for example, we can choose $\eta^-(\delta)=\sqrt{\delta}$ and $\eta^+(\delta)=2\sqrt{\delta}$). We also set $\nu=r-R_0$.

- Since $\eta(\delta)$ tends to $0$ when $\delta$ tends to $0$, for the far field terms, the overlapping area corresponds to $\nu\approx 0$. Since $u_n$ shall be regular, we can use its Taylor expansion in $\Omega_{M,\delta}$:

$$u_n^{\pm}(R_0,\theta)=\sum_{k\in\mathbb{N}}\frac{\nu^k}{k!}\frac{\partial^k u_n^{\pm}(R_0,\theta)}{\partial r^k}\;.$$

  Therefore in the overlapping area the far field is given by

$$\sum_{n\in\mathbb{N}}\delta^n u_n(r,\theta)=\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}\delta^n\frac{\nu^k}{k!}\frac{\partial^k u_n^{\pm}(R_0,\theta)}{\partial r^k}\;.\qquad(22)$$

- For the near field terms, since $\dfrac{\eta(\delta)}{\delta}$ tends to $\pm\infty$ when $\delta$ tends to $0$, the modal expansions of $U_n$ (20) and (21) can be used:

$$U_n(\theta, \frac{R_0\theta}{\delta}, \frac{\nu}{\delta}) = \sum_{k=0}^{n+1} C_{n,k}^{\pm}(\theta)\nu^k\delta^{-k} + o\left(\left(\frac{\nu}{\delta}\right)^{-\infty}\right).$$

Therefore, summing with respect to $n$ and reordering the terms give

$$\sum_{n\in\mathbb{N}} \delta^n U_n(\theta, \frac{R_0\theta}{\delta}, \frac{\nu}{\delta}) = \sum_{n=-1}^{+\infty}\sum_{k\in\mathbb{N}} \delta^n\nu^k C_{n+k,k}^{\pm} + o\left(\left(\frac{\nu}{\delta}\right)^{-\infty}\right). \tag{23}$$

Identifying the two series (23) and (22) and separating the powers of $\nu$ and $\delta$ give the matching conditions:

$$\boxed{C_{n,k}^{\pm} = \begin{cases} 0 & \text{if } k = n+1, \\ \dfrac{1}{k!}\dfrac{\partial^k u_{n-k}^{\pm}(R_0,\theta)}{\partial r^k} & \text{if } 0 \le k \le n. \end{cases}} \tag{24}$$

As explained in Appendix B, the matching conditions for $k \geq 2$ are redundant conditions. This is due to the fact that $U_n$ and $u_n^{\pm}$ are solutions to second order PDE. For instance knowing the Cauchy data $u_n^{\pm}(R_0,\theta)$ and $\frac{\partial u_n^{\pm}}{\partial r}(R_0,\theta)$ are sufficient to determine higher order derivatives of $u_n^{\pm}$. Also from expressions (20) and (21), one notices that after applying the Laplace operator to $U_n$ that the $C_{n,k}^{\pm}$ can be entirely determined for $k \geq 2$.
Let us also notice that these matching conditions provide transmission conditions for the far field terms $u_n$ since

$$[u_n] = C_{n,1}^{+} - C_{n,1}^{-} \qquad \text{and} \qquad \left[\frac{\partial u_n}{\partial r}\right] = C_{n,0}^{+} - C_{n,0}^{-}.$$

We shall prove in Appendix B that the system of equations made of the far fields equations (17), the near fields equations (19) and the matching conditions (24) entirely define the far fields terms $u_n$ and the near fields terms $U_n$ of the asymptotic expansion for any order $n$ (see also proposition 2.8). The proof uses an abstract framework, first introduced by Claeys [11], that may be hard to catch for readers who are not familiar with matched asymptotic expansions. This is why we found it useful to explicit the construction for the first terms using a step by step method. Theses terms are the ones used in the approximate transmission conditions later on.

## 2.2 Construction of the First Terms of the Asymptotic Expansion

In this section, we first compute step by step $u_0$, $u_1$ and $U_0$, $U_1$ and $U_2$. A preliminary important step consists in finding a suitable framework to solve near field problems (as explained in [18] (chapter 2), [19] and [8]).

### 2.2.1 Variational Formulation for the Resolution of Near Fields Problems

Let us consider the following problem for the unknown $U(\mathcal{V}, S)$ and a given data $F$:

$$\begin{cases} \nabla\cdot(\mu\nabla U) = F \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \\ U(\mathcal{V}, S+1) = U(\mathcal{V}, S). \end{cases} \tag{25}$$

We introduce the weighted-periodic functional space $W^1(\mathbb{R}^2)$

$$W_1(\mathbb{R}^2) := \left\{ v \in \mathcal{D}'(\mathbb{R}^2), \nabla v \in L^2(B_0), \frac{v}{(1+\mathcal{V}^2)^{\frac{1}{2}}} \in L^2(B_0) \text{ and } v(\cdot, S+1) = v(\cdot, S) \right\},$$

equipped with the scalar product

$$(U,V)_{W_1} := \int_{B_0} \nabla U \cdot \nabla \bar{V} \; dSd\mathcal{V} \; + \; \int_{B_0} \frac{1}{1+\mathcal{V}^2} U \bar{V} \; dSd\mathcal{V},$$

where $B_0$ is the periodicity cell (15).
Since constant functions are in $W_1(\mathbb{R}^2)$ and are solutions of (25), we also consider the quotient space

$$\mathcal{W}(\mathbb{R}^2) = W_1 | \mathbb{R}.$$

The norm on $\mathcal{W}(\mathbb{R}^2)$ is defined by

$$\|U\|_{\mathcal{W}} = \inf_{c \in \mathbb{C}} \|U + c\|_{W_1(\mathbb{R}^2)}.$$

The following proposition gives basic properties of $\mathcal{W}(\mathbb{R}^2)$. The proof is standard and is done is appendix A.3

**Proposition 2.3.** *$\mathcal{W}(\mathbb{R}^2)$ is a Hilbert space. Moreover the semi-norm $u \mapsto \left( \int_{B_0} |\nabla U|^2 dSd\mathcal{V} \right)^{1/2}$ is an equivalent norm on $\mathcal{W}(\mathbb{R}^2)$.*

We now introduce the variational formulation associated with the problem (25): find $U \in \mathcal{W}(\mathbb{R}^2)$ such that

$$a(U,V) = < F, V > \quad \forall \; V \in \mathcal{W}(\mathbb{R}^2), \tag{26}$$

where,

$$a(U,V) = \int_{B_0} \mu \nabla U \cdot \nabla \bar{V} \; dSd\mathcal{V},$$

and $<,>$ denotes the duality product on the dual $w\,(\mathbb{R}^2)^\star$ of $\mathcal{W}(\mathbb{R}^2)$ with respect to the $L^2$ duality product.
The following existence and uniqueness result is an immediate consequence of the Lax Milgram theorem:

**Proposition 2.4.** *Assume that $F \in \mathcal{W}(\mathbb{R}^2)^\star$. Problem* (26) *has a unique solution in $\mathcal{W}(\mathbb{R}^2)$. If we further assume that $\langle F, 1 \rangle_{W_1(\mathbb{R}^2)^\star, W_1(\mathbb{R}^2)} = 0$, then the solution to (26) satisfies (25).*

#### 2.2.2 Construction of the first near field terms : $U_0$, $U_1$ and $U_2$

In the remainder of this section, we assume that $u_0$, $u_1$ and $u_2$ exist. This assumption will be verified latter. With the previous framework it is possible to determine the $U_0$,$U_1$ and $U_2$. We also obtain semi-explicit formulas for these terms by separating the macroscopic variable $\theta$ form the microscopic variables $S$ and $\mathcal{V}$.

**• Construction of** $U_0$
Using the matching conditions (24) and the modal expansions (20) and (21), we introduce $U_0 \in W_1(\mathbb{R}^2)$ solution to

$$\mathcal{A}_0 U_0 = R_0^2 \nabla \cdot \mu \nabla U_0 = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^2)$$

From Proposition 2.4, we deduce that $U_0(\mathcal{V}, S, \theta) = cte(\theta)$. In addition the matching conditions tell us that

$$U_0(\mathcal{V}, S, \theta) = u_0^\pm(R_0, \theta) + o(\mathcal{V}^{-\infty}) \quad \text{for } \pm \mathcal{V} > \frac{1}{2}.$$

Therefore $u_0^+(R_0,\theta) = u_0^-(R_0,\theta) = cte(\theta)$ which means in particular that

$$[u_0] = 0.$$

Le us notice that we can also write that

$$U_0(\mathcal{V}, S, \theta) = \langle u_0 \rangle(\theta)$$

For convenience, we introduce the notation $V_0^0(\mathcal{V}, S) := 1$ so that $U_0(S,\mathcal{V},\theta) = \langle u_0(\theta)\rangle V_0^0(S,\mathcal{V})$. This complicated notation will be justified by subsequent formulas for higher order terms.
To shorten the notation, we shall no more indicate the dependence on $\theta$ since it plays only the role of a parameter in the following problems.

**• Construction of $U_1$**
From the expression of $U_0$ and (19), we observe that one needs to construct $U_1$ solution to the following equations:

$$\left\{\begin{array}{l} \nabla \cdot \mu \nabla U_1 = -\dfrac{1}{R_0^2}\left(\langle u_0 \rangle \mathcal{A}_1^0(V_0^0) + \langle \dfrac{\partial u_0}{\partial \theta}\rangle \mathcal{A}_1^\theta(V_0^0)\right) \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \\ U_1(\mathcal{V}, S+1, \mathcal{V}, \theta) = U_1(\mathcal{V}, S, \theta), \end{array}\right. \tag{27}$$

and which satisfies an asymptotic behaviour

$$U_1(\mathcal{V}, S, \theta) = u_1^\pm(R_0,\theta) + \mathcal{V}\frac{\partial u_0^\pm}{\partial r} + o(\mathcal{V}^{-\infty}) \quad \text{for } \pm\mathcal{V} > \frac{1}{2}. \tag{28}$$

We shall construct $U_1$ as

$$U_1(\mathcal{V}, S, \theta) = \tilde{U}_1(\mathcal{V}, S, \theta) + \chi(\mathcal{V}) P(\mathcal{V}, \theta), \tag{29}$$

where $\tilde{U}_1 \in W_1(\mathbb{R}^2)$ satisfies a problem of the form (26), P is a function which has a polynomial behaviour with respect to $\mathcal{V}$ when $\pm\mathcal{V} \to +\infty$ and $\chi$ is a smooth truncation function such that

$$\chi(\mathcal{V}) = \left\{\begin{array}{ll} 1 & \text{if } |\mathcal{V}| \geq 2, \\ 0 & \text{if } |\mathcal{V}| \leq 1. \end{array}\right.$$

**Remark 2.5.** *In this construction $\tilde{U}_1$ depends on $\chi$ but $U_1$ does not depend on this function. To see that we first observe that if we have constructed two functions $U_1$ and $U_1'$, then from (29) and (28) we see that the difference $D_1 = U_1 - U_1'$ is in $W_1(\mathbb{R}^2)$, and goes to $0$ as $|\mathcal{V}| \to \pm\infty$. Moreover equation (27) yields that $D_1$ satisfies (25) with $F = 0$. Then Proposition 2.4 implies $D_1$ is equal to $0$.*

Using the linearity with respect to $\langle \dfrac{\partial u_0}{\partial r}\rangle$ and $\langle \dfrac{\partial u_0}{\partial \theta}\rangle$, it will be useful to consider three canonical functions $W_0^0$, $V_1^0$ and $V_1^1$ solutions to the following problems:

- $V_1^0 \in W_1(\mathbb{R}^2)$ and

$$\left\{\begin{array}{l} \nabla \cdot (\mu \nabla V_1^0) = -\dfrac{1}{R_0^2}\mathcal{A}_1^0 V_0^0 \quad \text{in } B_0, \\ V_1^0 = \pm A_0^+(V_1^0) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} > \dfrac{1}{2}. \end{array}\right. \tag{30}$$

- $V_1^1 \in W_1(\mathbb{R}^2)$ and

$$\left\{\begin{array}{l} \nabla \cdot (\mu \nabla V_1^1) = -\dfrac{1}{R_0^2}\mathcal{A}_1^\theta V_0^0 = -\dfrac{1}{R_0} \quad \text{in } B_0, \\ V_1^1 = \pm A_0^+(V_1^1) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} > \dfrac{1}{2}. \end{array}\right. \tag{31}$$

- $W_0^0$ is such that $W_0^0 - \chi\mathcal{V} \in W_1(\mathbb{R}^2)$, and

$$\begin{cases} \nabla\cdot(\mu\nabla W_0^0) = 0 \quad \text{in } B_0, \\ W_0^0 = \pm A_0^+(W_0^0) + \mathcal{V} + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} > \dfrac{1}{2}. \end{cases} \tag{32}$$

In the equations above, $A_0^+(V)$ denotes a constant (with respect to $\mathcal{V}$ and $S$) that may depend on $V$: $A_0^+(V)$ is the coefficient of degree 0 in the polynomial behaviour of $V$ for $\mathcal{V} > \dfrac{1}{2}$.

**Proposition 2.6.** *Problems (31), (30) and (32) have a unique solution (of the form (29)). Moreover* $V_1^0 = 0$.

*Proof.*

- Since $\mathcal{A}_0^0 V_0^0 = 0$, Proposition 2.4 implies that $V_1^0$ is constant. The second equation in (30) implies that $V_1^0 = 0$.
- $\mathcal{A}_1^\theta(V_0^0) = -\frac{1}{R_0}\frac{\partial\mu}{\partial S}$. Moreover $\frac{\partial\mu}{\partial S} \in W_1(\mathbb{R}^2)^\star$. Proposition 2.4 proves that $V_1^1$ exists and is unique up to a constant. Proposition 2.1 and the second equation in (31) set this constant.
- $W_0^0$ does not have a variational behaviour for large $\mathcal{V}$. We therefore cannot directly apply Proposition 2.4. However note that if $W_0^0$ exists then it is unique (using similar arguments as for $U_1$).
We set $\tilde{W}_0^0 := W_0^0 - \chi(\mathcal{V})\mathcal{V}$. Then,

$$\begin{cases} \nabla\cdot(\mu\nabla \tilde{W}_0^0) = \tilde{f} \quad \text{in } B_0, \\ \tilde{W}_0^0 = \pm A_0^+(W_0^0) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} > \dfrac{1}{2}, \end{cases}$$

where $\tilde{f} := -\mu_\infty(\Delta(\chi(\mathcal{V})\mathcal{V})) = -\mu_\infty(2\chi'(\mathcal{V}) + \chi''(\mathcal{V})\mathcal{V})$.
Since $\tilde{f}$ is regular and is compactly-supported it is clear that $\tilde{f} \in W_1^1(\mathbb{R}^2)^\star$. To obtain the existence of $\tilde{W}_0^0$, we only need to check the compatibility condition $\displaystyle\int_{B_0} \tilde{f}\; dSd\mathcal{V} = 0$. But this is verified since,

$$\begin{aligned} \int_{B_0} \tilde{f}\; dSd\mathcal{V} &= \lim_{\mathcal{V}_0\to+\infty} \int_{-\mathcal{V}_0}^{\mathcal{V}_0}\int_{-\frac{1}{2}}^{\frac{1}{2}} -\mu_\infty \triangle(\chi(\mathcal{V})\mathcal{V})\; dSd\mathcal{V}, \\ &= \lim_{\mathcal{V}_0\to+\infty} \left( \frac{\partial(\chi(\mathcal{V})\mathcal{V})}{\partial\mathcal{V}}_{|\mathcal{V}=-\mathcal{V}_0} - \frac{\partial(\chi(\mathcal{V})\mathcal{V})}{\partial\mathcal{V}}_{|\mathcal{V}=\mathcal{V}_0} \right), \\ &= 0. \end{aligned}$$

□

Now, we can explicitly construct $U_1$ as

$$U_1(\mathcal{V},S,\theta) = \alpha(\theta)V_0^0(S,\mathcal{V}) \;+\; \beta(\theta)\; W_0^0(\mathcal{V},S) \;+\; \langle\frac{\partial u_0}{\partial\theta}(R_0,\theta)\rangle V_1^1(\mathcal{V},S).$$

where $\alpha(\theta)$ and $\beta(\theta)$ have to be determined. It is clear that $U_1$ satisfies the equation (27) since

$$\begin{aligned} \nabla\cdot(\mu\nabla\tilde{U}_1) &= \alpha(\theta)\underbrace{\nabla\cdot(\mu\nabla V_0^0)}_{=0} + \beta(\theta)\underbrace{\nabla\cdot(\mu\nabla W_0^0)}_{=0} + \langle\frac{\partial u_0}{\partial\theta}\rangle\nabla\cdot\mu\nabla V_1^1, \\ &= -\langle\frac{\partial u_0}{\partial\theta}\rangle\frac{1}{R_0}\frac{\partial\mu}{\partial S}. \end{aligned}$$

Now we are going to determine $\alpha$ and $\beta$. The behaviour of $U_1$ for large $\mathcal{V}$ (when $\mathcal{V} > \dfrac{1}{2}$) is given by

$$U_1 \quad \sim \quad \alpha(\theta) + \left( A_0^+(W_0^0)\beta(\theta) + A_0^+(V_1^1)\langle\frac{\partial u_0}{\partial\theta}\rangle \right) \;+\; \mathcal{V}\; A_1^+(W_0^0)\beta(\theta). \tag{33}$$

In the same way, when $\mathcal{V} < \dfrac{1}{2}$,

$$U_1 \quad \sim \quad \alpha(\theta) - \left( A_0^+(W_0^0)\beta(\theta) + A_0^+(V_1^1)\langle\frac{\partial u_0}{\partial\theta}\rangle \right) \;+\; \mathcal{V}\; A_1^+(W_0^0)\beta(\theta). \tag{34}$$

To determine $\alpha$ and $\beta$ it suffices to identify the different powers of $\mathcal{V}$ in (33),(34) with (28):

- The identification of the term of degree $1$ gives

  $$\frac{\partial u_0^+}{\partial r} = \beta(\theta) \qquad \text{and} \qquad \frac{\partial u_0^-}{\partial r} \;=\; \beta(\theta).$$

  Therefore, we obtain

  $$\left[\frac{\partial u_0}{\partial r}\right] = 0 \quad \text{and} \quad \beta(\theta) = \langle\frac{\partial u_0}{\partial r}\rangle.$$

- In the same manner the identification of the term of degree $0$ gives

  $$u_1^+ = \alpha(\theta) \;+\; A_0^+(W_0^0)\beta(\theta) \;+\; A_0^+(V_1^1)\langle\frac{\partial u_0}{\partial\theta}\rangle,$$
  $$u_1^- = \alpha(\theta) \;-\; A_0^+(W_0^0)\beta(\theta) \;-\; A_0^+(V_1^1)\langle\frac{\partial u_0}{\partial\theta}\rangle.$$

  Therefore, adding and subtracting the two previous equations yield

  $$[u_1] = 2\left( A_0^+(W_0^0)\langle\frac{\partial u_0}{\partial r}\rangle \;+\; A_0^+(V_1^1)\langle\frac{\partial u_0}{\partial\theta}\rangle \right) \quad \text{and} \quad \alpha(\theta) = \langle u_1\rangle.$$

To summarize,

$$\boxed{U_1 = \langle u_1\rangle V_0^0 \;+\; \langle\tfrac{\partial u_0}{\partial\theta}\rangle\; V_1^1 \;+\; \langle\tfrac{\partial u_0}{\partial r}\rangle\; W_0^0} \tag{35}$$

$$[u_1] = 2A_0^+(W_0^0)\langle\frac{\partial u_0}{\partial r}\rangle \;+\; 2A_0^+(V_1^1)\langle\frac{\partial u_0}{\partial\theta}\rangle \tag{36}$$

$$\left[\frac{\partial u_0}{\partial r}\right] = 0 \tag{37}$$

**• Construction of $U_2$**
One needs to construct $U_2$ 1-periodic in $S$ solution of the following equation

$$\nabla\cdot\mu\nabla U_2 = -\frac{1}{R_0^2}\left(\langle\frac{\partial u_1}{\partial\theta}\rangle\mathcal{A}_1^\theta(V_0^0) \;+\; \langle\frac{\partial^2 u_0}{\partial\theta^2}\rangle(\mathcal{A}_2^{\theta\theta}(V_0^0) \;+\; \mathcal{A}_1^\theta(V_1^1)) \right.$$
$$\left. +\;\langle\frac{\partial u_0}{\partial\theta}\rangle\mathcal{A}_1^0(V_1^1) + \langle u_0\rangle\mathcal{A}_2^0(V_0^0) \;+\; \langle\frac{\partial u_0}{\partial r}\rangle\mathcal{A}_1^0(W_0^0) + \langle\frac{\partial^2 u_0}{\partial r\partial\theta}\rangle\mathcal{A}_1^\theta(W_0^0)\right), \tag{38}$$

and which satisfies the following asymptotic behaviour

$$U_2(\mathcal{V},S,\theta) = u_2^{\pm} \; + \; \mathcal{V}\frac{\partial u_1^{\pm}}{\partial r} \; + \; \frac{\mathcal{V}^2}{2}\langle\frac{\partial^2 u_0}{\partial r^2}\rangle + o(\mathcal{V}^{-\infty}) \quad \text{for } \pm\mathcal{V} > \frac{1}{2}. \tag{39}$$

We adopt the same method as for $U_1$: we shall construct $U_2$ as

$$U_2(\mathcal{V},S,\theta) = \tilde{U}_2(\mathcal{V},S,\theta) + \chi(\mathcal{V})P^{+}(\mathcal{V},\theta) + \chi(-\mathcal{V})P^{-}(\mathcal{V},\theta), \tag{40}$$

where $\tilde{U}_2 \in W_1(\mathbb{R}^2)$ satisfies a problem of the form 26, $P^{\pm}$ are polynomials and $\chi$ is a smooth truncation function such that

$$\chi(\mathcal{V}) = \begin{cases} 1 & \text{if } \mathcal{V} \geq 2, \\ 0 & \text{if } \mathcal{V} \leq 1. \end{cases} \tag{41}$$

In order to separate the macroscopic variable $\theta$ from the microscopic ones, it will be useful to consider five new canonical functions solutions to the following problems:

- $V_2^0$ such that $V_2^0 - \chi(\mathcal{V})P^{+}(V_2^0) - \chi(-\mathcal{V})P^{-}(V_2^0) \in W_1(\mathbb{R}^2)$ and

$$\begin{cases} \nabla\cdot(\mu\nabla V_2^0) = -\dfrac{1}{R_0^2}\mathcal{A}_2^0(V_0^0) \quad \text{in } B_0, \\ V_2^0 = \pm A_0^{+}(V_2^0) + P^{\pm}(V_2^0)(\mathcal{V}) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \mathcal{V} > \dfrac{1}{2}, \end{cases} \tag{42}$$

  where,

$$P^{\pm}(V_2^0)(\mathcal{V}) = \pm\mathcal{V}A_1^{+}(V_2^0) + \frac{\mathcal{V}^2}{2}A_2^{\pm}(V_2^0),$$

  with,

$$A_2^{\pm}(V_2^0) = -\frac{\omega^2\rho_\infty}{\mu_\infty},$$

$$2A_1^{+}(V_2^0) = -\mathcal{V}_0(A_2^{+}(V_2^0) + A_2^{-}(V_2^0)) - \frac{1}{\mu_\infty R_0^2}\int_{-\mathcal{V}_0}^{\mathcal{V}_0}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{A}_2^0(V_0^0) \quad (|\mathcal{V}_0| > \frac{1}{2}).$$

- $V_2^1 \in W_1(\mathbb{R}^2)$ such that

$$\begin{cases} \nabla\cdot(\mu\nabla V_2^1) = -\dfrac{1}{R_0^2}\mathcal{A}_1^0(V_1^1) \quad \text{in } B_0, \\ V_2^1 = \pm A_0^{+}(V_2^1) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \mathcal{V} > \dfrac{1}{2}. \end{cases} \tag{43}$$

- $V_2^2$ such that $V_2^2 - \chi(\mathcal{V})P^{+}(V_2^2) - \chi(-\mathcal{V})P^{-}(V_2^2) \in W_1(\mathbb{R}^2)$ and

$$\begin{cases} \nabla\cdot(\mu\nabla V_2^2) = -\dfrac{1}{R_0^2}(\mathcal{A}_1^{\theta}(V_1^1) + \mathcal{A}_2^{\theta\theta}V_0^0) \quad \text{in } B_0, \\ V_2^2 = \pm A_0^{+}(V_2^2) + P^{\pm}(V_2^2)(\mathcal{V}) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \mathcal{V} > \dfrac{1}{2}, \end{cases} \tag{44}$$

  where,

$$P^{\pm}(V_2^2)(\mathcal{V}) = (\pm\mathcal{V})A_1^{+}(V_2^2) + \frac{\mathcal{V}^2}{2}A_2^{\pm}(V_2^2),$$

  with

$$A_2^{\pm}(V_2^2) = -\frac{1}{R_0^2},$$

$$2A_1^{+}(V_2^2) = -\mathcal{V}_0(A_2^{+}(V_2^2) + A_2^{-}(V_2^2)) - \frac{1}{\mu_\infty R_0^2}\int_{-\mathcal{V}_0}^{\mathcal{V}_0}\int_{-\frac{1}{2}}^{\frac{1}{2}}(\mathcal{A}_1^{\theta}(V_1^1) + \mathcal{A}_2^{\theta\theta}V_0^0) \quad (|\mathcal{V}_0| > \frac{1}{2}).$$

- $W_1^0$ such that $W_1^0 - \chi(\mathcal{V})P^+(W_1^0) - \chi(-\mathcal{V})P^-(W_1^0) \in W_1(\mathbb{R}^2)$ and

$$\begin{cases} \nabla \cdot (\mu \nabla W_1^0) = -\dfrac{1}{R_0^2}(\mathcal{A}_1^0(W_0^0)) \quad \text{in } B_0, \\ W_1^0 = \pm A_0^+(W_1^0) + P^\pm(W_1^0)(\mathcal{V}) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \mathcal{V} > \dfrac{1}{2}, \end{cases} \tag{45}$$

where,

$$P^\pm(W_1^0)(\mathcal{V}) = (\pm\mathcal{V})A_1^+(W_1^0) + \frac{\mathcal{V}^2}{2}A_2^\pm(W_1^0),$$

with,

$$A_2^\pm(W_1^0) = -\frac{1}{R_0},$$

$$2A_1^+(W_1^0) = -\mathcal{V}_0(A_2^+(W_1^0) + A_2^-(W_1^0)) - \frac{1}{\mu_\infty R_0^2}\int_{-\mathcal{V}_0}^{\mathcal{V}_0}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{A}_1^0(W_0^0) = 0 \quad (|\mathcal{V}_0| > \frac{1}{2}).$$

- $W_1^1 - (\chi(\mathcal{V}) - \chi(-\mathcal{V}))A_1^+(W_1^1)\mathcal{V} \in W_1(\mathbb{R}^2)$ and

$$\begin{cases} \nabla \cdot (\mu \nabla W_1^1) = -\dfrac{1}{R_0^2}(\mathcal{A}_1^\theta(W_0^0)) \quad \text{in } B_0, \\ W_1^1 = \pm A_0^+(W_1^1) + (\pm\mathcal{V})A_1^+(W_1^1) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \mathcal{V} > \dfrac{1}{2}, \end{cases} \tag{46}$$

where

$$2A_1^+(W_1^1) = -\frac{1}{\mu_\infty R_0^2}\int_{-\mathcal{V}_0}^{\mathcal{V}_0}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathcal{A}_1^\theta(W_0^0) \quad (|\mathcal{V}_0| > \frac{1}{2}).$$

Using the linearity of the equation (38), it is natural to construct $U_2$ as

$$\begin{aligned} U_2(\mathcal{V}, S, \theta) \quad = \quad & \alpha(\theta)V_0^0(\mathcal{V}, S) + \beta(\theta)W_0^0(\mathcal{V}, S) + \langle\frac{\partial u_1}{\partial\theta}\rangle V_1^1(\mathcal{V}, S) + \langle\frac{\partial^2 u_0}{\partial\theta^2}\rangle V_2^2(\mathcal{V}, S) + \langle\frac{\partial u_0}{\partial\theta}\rangle V_2^1(\mathcal{V}, S) \\ & + \langle u_0\rangle V_2^0(\mathcal{V}, S) + \langle\frac{\partial u_0}{\partial r}\rangle W_1^0(\mathcal{V}, S) + \langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle W_1^1(\mathcal{V}, S), \end{aligned} \tag{47}$$

where $\alpha(\theta)$ and $\beta(\theta)$ are two functions of $\theta$ that have to be determined.
By construction, it is clear that $U_2$ satisfies (38). To compute $\alpha(\theta)$ and $\beta(\theta)$, we have to identify the terms of order 0 and 1 of the polynomial expansion of $U_2$.

- The identification of the polynomial term of degree 1 gives

$$\frac{\partial u_1^+}{\partial r} = \beta(\theta) + \left(A_1^+(V_2^2)\langle\frac{\partial^2 u_0}{\partial\theta^2}\rangle + A_1^+(V_2^0)\langle u_0\rangle + A_1^+(W_1^1)\langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle\right),$$

$$\frac{\partial u_1^+}{\partial r} = \beta(\theta) - \left(A_1^+(V_2^2)\langle\frac{\partial^2 u_0}{\partial\theta^2}\rangle + A_1^+(V_2^0)\langle u_0\rangle. + A_1^+(W_1^1)\langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle\right).$$

Adding and subtracting the two previous equations give

$$\beta(\theta) = \langle\frac{\partial u_1}{\partial r}\rangle, \tag{48}$$

and

$$\left[\frac{\partial u_1}{\partial r}\right] = 2\ A_1^+(V_2^2)\ \langle\frac{\partial^2 u_0}{\partial\theta^2}\rangle\ +\ 2\ A_0^+(V_1^0)\ \langle u_0\rangle\ +\ 2\ A_1^+(W_1^1)\ \langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle.$$

- In the same manner, replacing $\beta$ by (48) in expression (47), the identification of the polynomial terms of degree 0 yields

$$u_2^+ = \alpha(\theta) + \left( A_0^+(W_0^0)\langle\frac{\partial u_1}{\partial r}\rangle + A_0^+(V_1^1)\langle\frac{\partial u_1}{\partial \theta}\rangle + A_0^+(V_2^2)\langle\frac{\partial^2 u_0}{\partial \theta^2}\rangle + A_0^+(V_2^1)\langle\frac{\partial u_0}{\partial \theta}\rangle \right.$$
$$\left. + A_0^+(V_2^0)\langle u_0\rangle + A_0^+(W_1^0)\langle\frac{\partial u_0}{\partial r}\rangle + A_0^+(W_1^1)\langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle \right),$$
$$u_2^- = \alpha(\theta) - \left( A_0^+(W_0^0)\langle\frac{\partial u_1}{\partial r}\rangle + A_0^+(V_1^1)\langle\frac{\partial u_1}{\partial \theta}\rangle + A_0^+(V_2^2)\langle\frac{\partial^2 u_0}{\partial \theta^2}\rangle + A_0^+(V_2^1)\langle\frac{\partial u_0}{\partial \theta}\rangle \right.$$
$$\left. + A_0^+(V_2^0)\langle u_0\rangle + A_0^+(W_1^0)\langle\frac{\partial u_0}{\partial r}\rangle + A_0^+(W_1^1)\langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle \right).$$

Adding and subtracting the two previous equations give

$$\alpha(\theta) = \langle u_2\rangle(\theta)$$

and

$$[u_2] = 2A_0^+(W_0^0)\langle\frac{\partial u_1}{\partial r}\rangle + 2A_0^+(V_1^1)\langle\frac{\partial u_1}{\partial \theta}\rangle + 2A_0^+(V_2^2)\langle\frac{\partial^2 u_0}{\partial \theta^2}\rangle + 2A_0^+(V_2^1)\langle\frac{\partial u_0}{\partial \theta}\rangle + 2A_0^+(V_2^0)\langle u_0\rangle$$
$$+ 2A_0^+(W_1^0)\langle\frac{\partial u_0}{\partial r}\rangle + 2A_0^+(W_1^1)\langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle.$$

- In addition, note that the identification of the polynomial term of degree 2 of $U_2$ is automatically true since

$$\underbrace{A_2^\pm(W_0^0)}_{0}\langle\frac{\partial u_1}{\partial r}\rangle + \underbrace{A_2^\pm(V_1^1)}_{0}\langle\frac{\partial u_1}{\partial \theta}\rangle + \underbrace{A_2^\pm(V_2^2)}_{-\frac{1}{R_0^2}}\langle\frac{\partial^2 u_0}{\partial \theta^2}\rangle + \underbrace{A_2^\pm(V_2^1)}_{0}\langle\frac{\partial u_0}{\partial \theta}\rangle$$
$$+ \underbrace{A_2^\pm(V_2^0)}_{-\frac{\omega^2\rho_\infty}{\mu_\infty}}\langle u_0\rangle + \underbrace{A_2^\pm(W_1^0)}_{-\frac{1}{R_0}}\langle\frac{\partial u_0}{\partial r}\rangle + \underbrace{A_2^\pm(W_1^1)}_{0}\langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle$$
$$= \langle\frac{\partial^2 u_0}{\partial r^2}\rangle.$$

To sum up:

$$\boxed{\begin{aligned} U_2(\mathcal{V}, S, \theta) \;=\;& \langle u_2\rangle\, V_0^0(\mathcal{V}, S) + \langle\frac{\partial u_1}{\partial r}\rangle\, W_0^0(\mathcal{V}, S) + \langle\frac{\partial u_1}{\partial \theta}\rangle\, V_1^1(\mathcal{V}, S) + \langle\frac{\partial^2 u_0}{\partial \theta^2}\rangle\, V_2^2(\mathcal{V}, S) \\ & + \langle\frac{\partial u_0}{\partial \theta}\rangle\, V_2^1(\mathcal{V}, S) + \langle u_0\rangle V_2^0(\mathcal{V}, S) + \langle\frac{\partial u_0}{\partial r}\rangle\, W_1^0(\mathcal{V}, S) + \langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle\, W_1^1(\mathcal{V}, S), \end{aligned}} \tag{49}$$

$$
\begin{aligned}
[u_2] \;&=\; 2A_0^+(W_0^0)\,\langle\frac{\partial u_1}{\partial r}\rangle \;+\; 2A_0^+(V_1^1)\,\langle\frac{\partial u_1}{\partial \theta}\rangle \;+\; 2A_0^+(V_2^2)\,\langle\frac{\partial^2 u_0}{\partial \theta^2}\rangle \;+\; 2A_0^+(V_2^1)\,\langle\frac{\partial u_0}{\partial \theta}\rangle \\
&\quad + 2A_0^+(V_2^0)\,\langle u_0\rangle \;+\; 2A_0^+(W_1^0)\,\langle\frac{\partial u_0}{\partial r}\rangle \;+\; 2A_0^+(W_1^1)\,\langle\frac{\partial^2 u_0}{\partial \theta \partial r}\rangle, \qquad (50)
\end{aligned}
$$

$$
\left[\frac{\partial u_1}{\partial r}\right] \;=\; 2A_1^+(V_2^2)\,\langle\frac{\partial^2 u_0}{\partial \theta^2}\rangle \;+\; 2A_1^+(V_2^1)\,\langle\frac{\partial u_0}{\partial \theta}\rangle \;+\; 2A_0^+(V_2^0)\,\langle u_0\rangle \;+\; 2A_1^+(W_1^1)\,\langle\frac{\partial^2 u_0}{\partial \theta \partial r}\rangle. \qquad (51)
$$

### 2.2.3 Construction of the First Far Field Terms: $u_0$ and $u_1$

In the previous part, we have seen that if $u_0$, $u_1$ and $u_2$ exist and are unique, thus $U_0$, $U_1$ and $U_2$ exist and are unique. Moreover the jumps and the normal derivative jumps of $u_1$ and $u_0$ across the interface $S_{R_0}$ are given by:

$$
\begin{cases}
[u_0] = 0, \\
\left[r\dfrac{\partial u_0}{\partial r}\right] = 0,
\end{cases}
\qquad \text{and} \qquad
\begin{cases}
[u_1] = A_0\langle r\dfrac{\partial u_0}{\partial r}\rangle + A_1\langle\dfrac{\partial u_0}{\partial \theta}\rangle, \\
\left[r\dfrac{\partial u_1}{\partial r}\right] = B_0\langle u_0\rangle + B_1\langle\dfrac{\partial^2 u_0}{\partial \theta \partial r}\rangle + B_2\langle\dfrac{\partial^2 u_0}{\partial \theta^2}\rangle,
\end{cases}
\qquad (52)
$$

where

$$
\begin{aligned}
&A_0 = \frac{2A_0^+(W_0^0)}{R_0}, \qquad A_1 = 2A_0^+(V_1^1), \\
&B_0 = 2R_0 A_1^+(V_2^0), \quad B_1 = 2R_0 A_1^+(W_1^0), \quad B_2 = 2R_0 A_1^+(V_2^2).
\end{aligned}
$$

From the previous conditions and the far fields term equations (17), it is possible to construct $u_0$ and $u_1$:

**• Construction of $u_0$**
Since $[u_0] = \left[r\dfrac{\partial u_0}{\partial r}\right] = 0$, we shall construct $u_0$ as solution to the following well-posed problem: find $u_0 \in H^1(\Omega)$ such that

$$
\boxed{
\begin{cases}
\triangle u_0 + \dfrac{\omega^2 \rho_\infty}{\mu_\infty} u_0 = \dfrac{f}{\mu_\infty} \quad \text{in } \mathcal{D}'(\Omega), \\
\dfrac{\partial u_0}{\partial r} + i\omega u_0 = 0 \text{ on } S_{R_e}.
\end{cases}
}
\qquad (53)
$$

Note that the limit problem is the problem without periodic ring.

**• Construction of $u_1$**
From (52) and (17), we shall construct $u_1$ as the solution to the following problem:
find $u_1 \in H^1(\Omega^+) \cap H^1(\Omega^-)$ such that

$$
\boxed{
\begin{cases}
\triangle u_1 + \dfrac{\omega^2 \rho_\infty}{\mu_\infty} u_1 = \dfrac{f}{\mu_\infty} \quad \text{in } \mathcal{D}'(\Omega^+) \cap \mathcal{D}'(\Omega^-), \\
\dfrac{\partial u_1}{\partial r} + i\omega u_1 = 0 \text{ on } S_{R_e}, \\
[u_1] = A_0\langle r\dfrac{\partial u_0}{\partial r}\rangle + A_1\langle\dfrac{\partial u_0}{\partial \theta}\rangle, \\
\left[r\dfrac{\partial u_1}{\partial r}\right] = B_0\langle u_0\rangle + B_1\langle\dfrac{\partial^2 u_0}{\partial \theta \partial r}\rangle + B_2\langle\dfrac{\partial^2 u_0}{\partial \theta^2}\rangle.
\end{cases}
}
\qquad (54)
$$

**Proposition 2.7.** *Problem (54) is well-posed.*

*Proof.* We first remark that if $u_1$ exists $u_1$ is unique. Then we shall construct $u_1$ as

$$u_1 = \tilde{u}_1 + \mathcal{R},$$

where $\tilde{u}_1 \in H^1(\Omega)$ and $\mathcal{R}$ is a bearing of the jump condition defined by

$$\mathcal{R} = \begin{cases} \chi(r - R_0)\left(A_0\langle r\frac{\partial u_0}{\partial r}\rangle + A_1\langle \frac{\partial u_0}{\partial \theta}\rangle\right) & \text{if } r > R_0, \\ 0 \quad \text{if } r < R_0, \end{cases}$$

and $\chi$ is a smooth function such that

$$\chi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \dfrac{R_e - R_0}{4}, \\ 0 & \text{if } x > \dfrac{R_e - R_0}{2}. \end{cases}$$

Note that, by construction $[\mathcal{R}] = [u_1]$. Since $u_0$ is smooth in the vicinity of $S_{R_0}$, $\mathcal{R}$ is in $H^1(\Omega^+) \cap H^1(\Omega^-)$. Moreover, since $\chi(r - R_0)$ is constant in the vicinity of $r = R_0$, $\left[r\dfrac{\partial \mathcal{R}}{\partial r}\right] = 0$.
It follows that $\tilde{u}_1$ satisfies the following classical variational form

$$\int_{\Omega^+\cup\Omega^-}\left(\nabla\tilde{u}_1 \cdot \nabla\bar{v} - \frac{\omega^2\rho_\infty}{\mu_\infty}\tilde{u}_1\bar{v}\right) + \int_{S_{R_e}} i\omega\tilde{u}_1\bar{v} = L(v) \qquad \forall v \in H^1(\Omega),$$

where,

$$\begin{aligned} L(v) \quad := \quad & \int_{\Omega^+\cup\Omega^-}\left(\triangle + \frac{\omega^2\rho_\infty}{\mu_\infty}\right)\mathcal{R}\bar{v} + \int_0^{2\pi}\left(B_0\langle u_0\rangle + B_1\langle\frac{\partial^2 u_0}{\partial\theta\partial r}\rangle + B_2\langle\frac{\partial^2 u_0}{\partial\theta^2}\rangle\right)\langle\bar{v}\rangle \\ & + \int_0^{2\pi}\underbrace{\left[r\frac{\partial\mathcal{R}}{\partial r}\right]}_{=0}\langle\bar{v}\rangle. \end{aligned}$$

Since the linear form $L$ is continue, ($|L(v)| \leq C\|v\|_{H^1(\Omega)}$), $\tilde{u}_1$ exists and is unique. Consequently, problem (54) is well-posed. ☐

## 2.3 Construction of the Whole Asymptotic Expansion

We can generalize the previous used approach to construct $u_n$ and $U_n$ for any $n$:

**Proposition 2.8.** *The system of equations made of (17),(19) and (24) has unique solutions $(u_n, U_n)$ such that $u_n \in H^1(\Omega^+) \cap H^1(\Omega^-)$, $U_n(\cdot,\cdot,\theta) \in H^1_{loc}(\mathbb{R}^2)$ and is non-exponentially increasing with respect to $\mathcal{V}$. Moreover,*

$$\begin{cases} \triangle u_n + \dfrac{\omega^2\rho_\infty}{\mu_\infty}u_n = \delta_n^0\dfrac{f}{\mu_\infty} \quad \text{in } \Omega^+\cup\Omega^-, \\ [u_n] = \displaystyle\sum_{j=1}^{n}\sum_{k=0}^{j} 2\langle\frac{\partial^k u_{n-j}}{\partial\theta^k}\rangle A_0^+(V_j^k) + \sum_{j=0}^{n-1}\sum_{k=0}^{j} 2\langle\frac{\partial^{k+1}u_{n-1-j}}{\partial\theta^k\partial r}\rangle A_0^+(W_j^k) \quad (a), \\ \left[\dfrac{\partial u_n}{\partial r}\right] = \displaystyle\sum_{j=2}^{n+1}\sum_{k=0}^{j} 2\langle\frac{\partial^k u_{n+1-j}}{\partial\theta^k}\rangle A_1^+(V_j^k) + \sum_{j=1}^{n}\sum_{k=0}^{j} 2\langle\frac{\partial^{k+1}u_{n-j}}{\partial\theta^k\partial r}\rangle A_1^+(W_j^k) \quad (b), \end{cases} \tag{55}$$

*and*

$$U_n(\mathcal{V}, S, \theta) = \sum_{j=0}^{n} \sum_{k=0}^{j} \langle \frac{\partial^k u_{n-j}(R_0, \theta)}{\partial \theta^k} \rangle V_j^k + \sum_{j=0}^{n-1} \sum_{k=0}^{j} \langle \frac{\partial^{k+1} u_{n-1-j}(R_0, \theta)}{\partial \theta^k \partial r} \rangle W_j^k, \tag{56}$$

*where* $(W_n^k)_{n\in\mathbb{N},k\leq n}$ *and* $(V_n^k)_{n\in\mathbb{N},k\leq n}$ *are two families of functions defined by (130) and (132) in the Appendix B . These functions depend only of the fast variables* $S$ *and* $\mathcal{V}$ *and are solutions of periodic cell problems posed in* $B_0$*.* $A_0^\pm$ *and* $A_1^\pm$ *are linear forms also defined in (130) and (132).*

The proof of this proposition is technical and is explained in details in the Appendix B. It is based on a new version of the matching conditions introduced by X.Claeys ([11]) and on the introduction of the two families of functions $(W_n^k)_{n\in\mathbb{N},k\leq n}$ and $(V_n^k)_{n\in\mathbb{N},k\leq n}$. These two families of functions can be understood as 'basis' functions for near fields terms.

**Remark 2.9.** *By writing this system, we have decoupled the computation of far fields terms from the computations of near fields terms. More precisely, we can first compute the far fields terms, and next, by post-processing, we can reconstruct the near fields terms.*

## 2.4 Justification of the Asymptotic Expansion

The justification of asymptotic expansion is classical. It is based on a the stability result of the exact problem (8) and a consistency result (Standard works on this kind of justification are [13] and [16]).

Again, we will denote by $\eta$ a smooth function such that

$$\lim_{\delta\to 0} \eta = 0, \quad \text{and} \quad \lim_{\delta\to 0} \frac{\eta}{\delta} = +\infty.$$

We also define a truncated function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\chi(x) = \begin{cases} 1 \text{ when } |x| \geq 1, \\ 0 \text{ when } |x| \leq 2. \end{cases}$$

We finally introduce $\chi_\eta(r,\theta) = \chi(\dfrac{r - R_0}{\eta})$ and $\varepsilon_n^\delta$, the error of order $n$

$$\varepsilon_n^\delta = u^\delta + (1 - \chi_\eta) u_e^n + \chi_\eta U_i^n, \tag{57}$$

where $u_e^n$ is the order $n$ truncated far field expansion and $U_i^n$ is the order $n$ truncated near field expansion:

$$u_e^n = \begin{cases} \sum_{i=0}^{n} \delta^i u_i^+ \text{ if } r > R_0, \\ \sum_{i=0}^{n} \delta^i u_i^- \text{ if } r < R_0, \end{cases}$$

$$U_i^n = \sum_{i=0}^{n} \delta^i U_i.$$

### 2.4.1 Consistency

**Proposition 2.10.** *There is a constant* $C$ *independent of* $\delta$ *such that*

$$|a^\delta(\varepsilon_n^\delta, v)| \leq C(\eta^{n-\frac{1}{2}} + \delta^{n-1}) \|v\|_{H^1(\Omega)}. \tag{58}$$

The proof is rather classical and is given in Annexe A.4. The main idea of the proof is to separate the consistency error $|a^\delta(\varepsilon_n^\delta, v)|$ into two parts: the matching error and the error on the Helmholtz equation in the near field area. Indeed, it is easily seen that

$$a^\delta(\varepsilon_n^\delta, v) = \underbrace{\int_\Omega \mu^\delta (u_e^n - U_i^n)\nabla\chi_\eta \cdot \nabla\bar{v} - \mu^\delta(\nabla(u_e^n - U_i^n)\cdot\nabla\chi_\eta)\bar{v}}_{\text{matching error}} + \underbrace{a(U_i^n, \chi_\eta v)}_{\text{equation error}} .$$

Estimating separately the two kinds of error gives the desired result (58). The upper bound of the matching error brings into play the matching conditions. To estimate $|a(U_i^n, \chi_\eta v)|$, we use that fact that by construction $U_i^n$ almost satisfies the Helmholtz equation.

### 2.4.2 Convergence

The previous consistency result (58) and the stability result (8) lead to the first convergence result:

$$\|\varepsilon^n\|_{H^1(\Omega)} \le C\eta^{n-1}. \tag{59}$$

This previous error estimate is not optimal. By triangular inequality, we can obtain better local error estimates:

**Proposition 2.11.** *Let us introduce*

$$\begin{aligned} \Omega_\gamma^- &= \left\{(x,y)\in R^2,\ \sqrt{x^2+y^2} < R_0 - \gamma\right\}, \\ \Omega_\gamma^+ &= \left\{(x,y)\in R^2,\ \sqrt{x^2+y^2} > R_0 + \gamma\right\}, \\ \Omega_\gamma &= \{(\mathcal{V},\theta)\in R\times[0,2\pi],\ -\gamma < \nu < +\gamma\}. \end{aligned} \tag{60}$$

*For any $n \ge 0$, for any $\gamma > 0$, there are a constant $C$ independent of $\delta$ and a constant $\delta_0 > 0$ such that for any $\delta < 0$,*

$$\|u^\delta - \sum_{l=0}^n \delta^l u_l^-\|_{H^1(\Omega_\gamma^-)} + \|u^\delta - \sum_{l=0}^n \delta^l u_l^+\|_{H^1(\Omega_\gamma^+)} \le C.\delta^{n+1} \tag{61}$$

*Similarly, for any $n \ge 0$, for any $\gamma > 0$ , there are a constant $C$ independent of $\delta$ and a constant $\delta_0 > 0$ such that for any $\delta < 0, \forall j \in \mathbb{N}$*

$$\begin{aligned} &\left(\int_{-\gamma}^{\gamma}\int_j^{j+1}\left|u^\delta{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})} - \sum_{k=0}^n \delta^k U_{k(\mathcal{V},S,\frac{S\delta}{R_0})}\right|^2 dS\ d\mathcal{V}\right)^{\frac12} \le C\delta^{n+1}, \\ &\left(\int_{-\gamma}^{\gamma}\int_j^{j+1}\left|\frac{\partial}{\partial\mathcal{V}}\left(u^\delta{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})} - \sum_{k=0}^n \delta^k U_{k(\mathcal{V},S,\frac{S\delta}{R_0})}\right)\right|^2 dS\ d\mathcal{V}\right)^{\frac12} \le C\delta^{n+1}, \\ &\left(\int_{-\gamma}^{\gamma}\int_j^{j+1}\left|\frac{\partial}{\partial S}\left(u^\delta{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})} - \sum_{k=0}^n \delta^k U_{k(\mathcal{V},S,\frac{S\delta}{R_0})}\right)\right|^2 dS\ d\mathcal{V}\right)^{\frac12} \le C\delta^{n+1}, \end{aligned} \tag{62}$$

$$\left(\int_{-\gamma}^{\gamma}\int_0^{2\pi}\left|u^\delta{}_{(R_0+\delta\mathcal{V},\theta)} - \sum_{k=0}^n \delta^k U_{k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)}\right|^2 d\theta\ d\mathcal{V}\right)^{\frac12} \le C\delta^{n+1},$$

$$\left(\int_{-\gamma}^{\gamma}\int_0^{2\pi}\left|\frac{\partial}{\partial\mathcal{V}}\left(u^\delta{}_{(R_0+\delta\mathcal{V},\theta)} - \sum_{k=0}^n \delta^k U_{k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)}\right)\right|^2 d\theta\ d\mathcal{V}\right)^{\frac12} \le C\delta^{n+1}.$$

*Proof.* By triangular inequality,

$$\|u^\delta - \sum_{l=0}^{n} \delta^l u_l^+\|_{H^1(\Omega_\gamma^+)} \leq \|u^\delta - \sum_{l=0}^{n+3} \delta^l u_l^+\|_{H^1(\Omega_\gamma^+)} + \underbrace{\|\sum_{l=n+1}^{n+3} \delta^l u_l^+\|_{H^1(\Omega_\gamma^+)}}_{\leq C\delta^{n+1}}.$$

But, for $\delta$ small enough, $u^\delta - \sum_{l=0}^{n+3} \delta^l u_l^+ = \varepsilon^{n+3}$ (57). Consequently,

$$\|u^\delta - \sum_{l=0}^{n} \delta^l u_l^+\|_{H^1(\Omega_\gamma^+)} \leq C(\eta(\delta)^{n+2} + \delta^{n+1}).$$

Choosing $\eta(\delta) = \delta^{\frac{n+1}{n+2}}$ leads to convergence estimate (61). The proof for (62) is similar and is given in Appendix (A.5). □

**Remark 2.12.** *It is possible to obtain a global result convergence estimate using multiscale expansion instead of matched asymptotic expansion (see for example [20] for a detailed comparison between matched asymptotic expansion and multiscale expansion): this expansion is given by*

$$\tilde{u}_\delta^n = \sum_{l=0}^{n} \delta^l \left( v_l^-(r,\theta) + v_l^+(r,\theta) + V_l(\frac{r-R_0}{\delta}, R_0\frac{\theta}{\delta}, \theta) \right),$$

*where*

$$\begin{cases} v_n = u_n \quad \forall n \in \mathbb{N}, \\ V_n = U_n - \mathbb{1}(\mathcal{V} > 0) \displaystyle\sum_{l=0}^{n} \frac{\mathcal{V}^l}{l!} \frac{\partial^l u_{n-l}^+(R_0,\theta)}{\partial r^l} - \mathbb{1}(\mathcal{V} < 0) \sum_{l=0}^{n} \frac{\mathcal{V}^l}{l!} \frac{\partial^l u_{n-l}^+(R_0,\theta)}{\partial r^l}. \end{cases}$$

# 3 Approximate Conditions

We are now in a position to build approximate models. These models are of course based on the asymptotic expansion. The main idea is to look for $u_n^\delta$ which is close to the first $n$ terms of the far fields expansion $\sum_{k=0}^{n} \delta^k u_k$ and which is solution of a variational problem. (see for instance [2] and [7]) .

The main difficulties in carrying out the construction of approximate model is to find well-posed variational approximate problems. For the moment, we do not have systematic method to derive approximate well-posed problem at any order. In this report, we only build the first and second order approximate conditions.

We adopt the following process to obtain approximate problems:

- First, we look for a variational problem: these problems are convenient to use finite-element methods.
- Then, in order to prove existence and uniqueness of the solution, we try to build approximate problems that satisfy the Fredholm alternative.
- Finally, we prove a uniform stability result with respect to $\delta$.

This method is already studied by [2],[3], [7] and [21] in the case of the derivation of effective boundary conditions. The case of effective transmission conditions modelling highly conductive thin sheets is treated by K.Schmidt [10].

## 3.1 First Order Approximate Problem

### 3.1.1 Building of the Approximate Problem

To manipulate simple expressions, we shall consider the special case where $\mu$ and $\rho$ are symmetric:

$$\left\{ \begin{array}{l} \mu(S,\mathcal{V}) = \mu(-S,\mathcal{V}) \quad \text{and} \quad \mu(S,\mathcal{V}) = \mu(S,-\mathcal{V}), \\ \rho(S,\mathcal{V}) = \rho(-S,\mathcal{V}) \quad \text{and} \quad \rho(S,\mathcal{V}) = \rho(S,-\mathcal{V}). \end{array} \right.$$

The general case do not present any additional difficulty, it is given in the appendix C.

**General Method**

Combining the result of the proposition 2.8 and the properties of the families $(V_n^k)$ and $(W_n^k)$ in the symmetric case (see appendix E) gives the problems verified by $u_0$ and $u_1$:

$$\begin{cases} \Delta u_0 + \omega^2 \dfrac{\rho_\infty}{\mu_\infty} u_0 = \dfrac{f}{\mu_\infty} \quad \text{in } \Omega^+ \cup \Omega^-, \\ [u_0] = 0 \quad \text{on } S_{R_0}, \\ \left[ r \dfrac{\partial u_0}{\partial r} \right] = 0 \quad \text{on } S_{R_0}, \\ \dfrac{\partial u_0}{\partial r} + i\omega u_0 = 0, \quad \text{on } S_{R_e} \end{cases} \quad \text{and} \quad \begin{cases} \Delta u_1 + \omega^2 \dfrac{\rho_\infty}{\mu_\infty} u_1 = 0 \quad \text{in } \Omega^+ \cup \Omega^-, \\ [u_1] = A_0 \langle r \dfrac{\partial u_0}{\partial r} \rangle \quad \text{on } S_{R_0}, \\ \left[ r \dfrac{\partial u_1}{\partial r} \right] = B_0 \langle u_0 \rangle + B_2 \langle \dfrac{\partial^2 u_0}{\partial \theta^2} \rangle \quad \text{on } S_{R_0}, \\ \dfrac{\partial u_1}{\partial r} + i\omega u_1 = 0 \quad \text{on } S_{R_e}, \end{cases}$$

where

$$A_0 = \frac{2A^+_{W_0^0,0}}{R_0} = -\frac{1}{R_0} - \frac{1}{R_0}\int_{-1/2}^{1/2}\left(W_0^0(\frac{1}{2},S) - W_0^0(-\frac{1}{2},S)\right)dS,$$

$$B_0 = 2R_0 A^+_{V_2^0,1} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{R_0\omega^2(\rho_\infty - \rho)}{\mu_\infty}dSd\mathcal{V},$$

$$B_2 = 2R_0 A^+_{V_2^2,1} = \frac{1}{\mu_\infty}\left(+\frac{\mu_\infty}{R_0} - \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\mu\left(\frac{1}{R_0} + \frac{\partial V_1^1}{\partial S}\right)dSd\mathcal{V}\right).$$

To compute $A_0$, $B_0$ and $B_2$, it suffices to compute the periodic cell problems satisfied by $V_1^1$ (31) and $W_0^0$ (32).

**Remark 3.1** (the case of constant coefficients)**.**
*If we assume that $\mu^\delta$ and $\rho^\delta$ are piecewise constants, which means that*

$$\mu^\delta = \begin{cases}\mu_0 & \textit{if } |\nu| \le \dfrac{\delta}{2},\\ \mu_\infty & \textit{else},\end{cases} \qquad \textit{and} \qquad \rho^\delta = \begin{cases}\rho_0 & \textit{if } |\nu| \le \dfrac{\delta}{2},\\ \rho_\infty & \textit{else},\end{cases}$$

*then the constants $A_0$, $B_0$ and $B_2$ have explicit expressions:*

$$A_0 = \frac{\mu_\infty - \mu_0}{\mu_0 R_0}, \qquad B_0 = \frac{R_0\omega^2}{\mu_\infty}(\rho_\infty - \rho_0) \quad \textit{and} \quad B_2 = \frac{\mu_\infty - \mu_0}{\mu_\infty R_0}.$$

Set $\tilde{u}_1^\delta = u_0 + \delta u_1$. Using the two previous problems, $\tilde{u}_1^\delta$ has the following properties:

$$\begin{cases}\Delta\tilde{u}_1^\delta + \omega^2\dfrac{\rho_\infty}{\mu_\infty}\tilde{u}_1^\delta = \dfrac{f}{\mu_\infty} \quad \text{in } \Omega^+\cup\Omega^-,\\ \left[\tilde{u}_1^\delta\right] = \delta A_0\langle r\dfrac{\partial\tilde{u}_1^\delta}{\partial r}\rangle + O(\delta^2) \quad \text{on } S_{R_0},\ (a)\\ \left[r\dfrac{\partial\tilde{u}_1^\delta}{\partial r}\right] = \delta\left(B_0\langle\tilde{u}_1^\delta\rangle + B_2\langle\dfrac{\partial^2\tilde{u}_1^\delta}{\partial\theta^2}\rangle\right) + O(\delta^2) \quad \text{on } S_{R_0},\ (b)\\ \dfrac{\partial\tilde{u}_1^\delta}{\partial r} + i\omega\tilde{u}_1^\delta = 0 \quad \text{on } S_{R_e}.\end{cases} \tag{63}$$

Consequently, it is natural to construct the first order approximate solution $\tilde{v}^\delta$ in the variational space $\tilde{\mathcal{V}}$ and satisfying $(\tilde{\mathcal{P}}_1)$:

$$\tilde{\mathcal{V}} = \left\{v \in H^1(\Omega^+)\cap H^1(\Omega^-) \text{ such that } \langle v(R_0,\theta)\rangle \in H^1_{per}(]0,2\pi[)\right\},$$

where,

$$H^1_{per}(]0,2\pi[) = \left\{v \in \mathcal{D}'(\mathbb{R}) \text{ such that } v \text{ is 1-periodic in } S \text{ and } \int_0^{2\pi}\left(|v'|^2 + |v|^2\right) < +\infty\right\},$$

and

$$(\tilde{\mathcal{P}}_1) \qquad \begin{cases}\Delta\tilde{v}_1^\delta + \omega^2\dfrac{\rho_\infty}{\mu_\infty}\tilde{v}_1^\delta = \dfrac{f}{\mu_\infty} \quad \text{in } \Omega^+\cup\Omega^-,\\ \left[\tilde{v}_1^\delta\right] = \delta A_0\langle r\dfrac{\partial\tilde{v}_1^\delta}{\partial r}\rangle \quad \text{on } S_{R_0},\\ \left[r\dfrac{\partial\tilde{v}_1^\delta}{\partial r}\right] = \delta\left(B_0\langle\tilde{v}_1^\delta\rangle + B_2\langle\dfrac{\partial^2\tilde{v}_1^\delta}{\partial\theta^2}\rangle\right) \quad \text{on } S_{R_0},\\ \dfrac{\partial\tilde{v}_1^\delta}{\partial r} + i\omega\tilde{v}_1^\delta = 0 \quad \text{on } S_{R_e}.\end{cases}$$

**Difficulties**

We consider the variational formulation associated to $(\tilde{\mathcal{P}}_1)$: find $\tilde{v}_1^\delta \in \tilde{\mathcal{V}}$ such that

$$\tilde{a}^\delta(\tilde{v}_1^\delta\, v) = \int_{\Omega^+\cup\Omega^-} \frac{f}{\mu_\infty}\bar{v} \quad \forall\, v \in \tilde{\mathcal{V}},$$

where,

$$\tilde{a}^\delta(u,v) = \int_{\Omega^+\cup\Omega^-}\left(\nabla u\cdot\nabla\bar{v} - \frac{\omega^2\rho_\infty}{\mu_\infty}u\bar{v}\right) + i\omega\int_{S_{R_e}} u\bar{v}$$
$$+\,\delta B_0\int_0^{2\pi}\langle u\rangle\langle\bar{v}\rangle - \delta B_2\int_0^{2\pi}\langle\frac{\partial u}{\partial\theta}\rangle\langle\frac{\partial\bar{v}}{\partial\theta}\rangle + \frac{1}{A_0\delta}\int_0^{2\pi}[u][\bar{v}].$$

It is clear that $\tilde{a}$ does not always satisfies the Fredholm alternative: although the first terms are compact or coercive, the term $\delta B_2\displaystyle\int_0^{2\pi}\langle\frac{\partial u}{\partial\theta}\rangle\langle\frac{\partial\bar{v}}{\partial\theta}\rangle$ can be neither coercive neither compact if $B_2 > 0$. Moreover, whereas $\displaystyle\frac{1}{A_0\delta}\int_0^{2\pi}[u][\bar{v}]$ is compact for a given $\delta$, this term cannot be easily controlled when $\delta$ tends to $0$.

**Remark 3.2.** *Note that this problem already exists in the case of homogeneous thin layers: if $\mu_\infty > \mu_0$ $A_0$ and $B_0$ are positive. If $\mu_\infty < \mu_0$ $A_0$ and $B_0$ are negative.*

### 3.1.2 Centered Approximate Problem

To overcome this difficulty, we do a consistent modification such that the two last terms becomes coercive. The main idea is to shift the jump terms of $\alpha\delta$ where $\alpha$ is a positive parameter that we have to determine later. Instead of computing the jump directly on the limit interface $S_{R_0}$,we will compute the jump between two different interfaces separated by $2\alpha\delta$ (see Fig. 5).

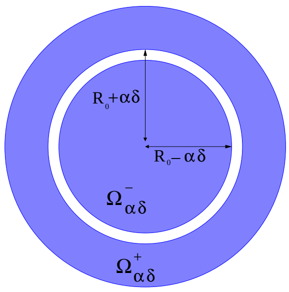



Figure 5: $\Omega_{\alpha\delta} = \Omega^+_{\alpha\delta}\cup\Omega^-_{\alpha\delta}$

For simplicity of notation, we introduce the following notation

$$g_\alpha^\pm := g(R_0+\alpha\delta),$$
$$\langle g\rangle_\alpha := \frac{1}{2}(g_\alpha^+ + g_\alpha^-),$$
$$[g]_\alpha := g_\alpha^+ - g_\alpha^-.$$

Combining Taylor expansions and (63-((a)-(b))) yields:

$$[\tilde{u}_1^\delta]_\alpha = \delta\,(A_0 + \frac{2\alpha}{R_0})\,\langle r\frac{\partial\tilde{u}_1^\delta}{\partial r}\rangle \;+\; O(\delta^2),$$

and

$$\left[r\frac{\tilde{u}_1^\delta}{\partial r}\right]_\alpha = \delta\left((B_0 - 2\alpha R_0 \frac{\omega^2\rho_\infty}{\mu_\infty})\langle\tilde{u}_1^\delta\rangle_\alpha \;+\; \left(B_2 - \frac{2\alpha}{R_0}\right)\langle\frac{\partial^2\tilde{u}_1^\delta}{\partial\theta^2}\rangle_\alpha\right) + O(\delta^2).$$

Let us introduce the constants $A_0^\alpha$, $B_0^\alpha$, $B_2^\alpha$ and the open sets $\Omega_{\alpha\delta}^\pm$

$$A_0^\alpha := A_0 + \frac{2\alpha}{R_0}, \qquad B_0^\alpha := B_0 - 2\alpha R_0 \frac{\omega^2\rho_\infty}{\mu_\infty}, \qquad B_2^\alpha := B_2 - \frac{2\alpha}{R_0},$$

$$\Omega_{\alpha\delta}^+ := \left\{(x,y)\in\mathbb{R}^2, R_0 + \alpha\delta < \sqrt{x^2+y^2} < R_e\right\},$$

$$\Omega_{\alpha\delta}^- := \left\{(x,y)\in\mathbb{R}^2, \sqrt{x^2+y^2} < R_0 - \alpha\delta\right\}.$$

The important point here is that there exists $\alpha^\star$ such that, for any $\alpha > \alpha^\star$, $B_0^\alpha$ is negative and $A_0^\alpha$ is positive.

**Remark 3.3.** *In the thin layer case,* $\alpha = \dfrac{1}{2}$ *works for any* $(\mu_0, \rho_0) \in (\mathbb{R}_\star^+)^2$.

To do the numerical analysis of this shifted approximate condition, we consider the Hilbert space,

$$V_{\alpha\delta} = \left\{u \in H^1(\Omega_{\alpha\delta}^+) \cap H^1(\Omega_{\alpha\delta}^-), \text{ such that } \langle u\rangle_\alpha \in H^1_{per}(]0,2\pi[)\right\},$$

equipped with the norm

$$\|v\|_{V_{\alpha\delta}} = \|v\|^2_{H_1(\Omega_{\alpha\delta}^+)} + \|v\|^2_{H_1(\Omega_{\alpha\delta}^-)} + \frac{1}{\delta A_0^\alpha}\int_0^{2\pi} |[v(\theta)]_\alpha|^2\, d\theta + \delta\,|B_2^\alpha|\,\mu_\infty \int_0^{2\pi}\left(|\langle v\rangle|^2 + \left|\langle\frac{\partial v}{\partial\theta}\rangle\right|^2\right) d\theta.$$

It is now clear that for $\alpha > \alpha^\star$ the problem, find $u_1^\delta \in V_{\alpha\delta}$

$$\begin{cases} \Delta u_1^\delta + \omega^2 \dfrac{\rho_\infty}{\mu_\infty} u_1^\delta = \dfrac{f}{\mu_\infty} \quad \text{in } \Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-, \\ \left[u_1^\delta\right]_\alpha = \delta A_0^\alpha \langle r\dfrac{\partial u_1^\delta}{\partial r}\rangle_\alpha \quad (a), \\ \left[r\dfrac{\partial u_1^\delta}{\partial r}\right]_\alpha = \delta\left(B_0^\alpha\langle u_1^\delta\rangle_\alpha + B_2^\alpha\langle\dfrac{\partial^2 u_1^\delta}{\partial\theta^2}\rangle_\alpha\right), \quad (b) \\ \dfrac{\partial u_1^\delta}{\partial r} + i\omega u_1^\delta = 0 \quad \text{on } S_{R_e}, \end{cases} \tag{64}$$

satisfies the Fredholm alternative if $\alpha > \alpha^\star$. Indeed, if we also consider the associated variational formulation, find $u_1^\delta \in V_{\alpha\delta}$ such that

$$a^\delta(u_1^\delta, v) = -\int_{\Omega_{\alpha\delta}^+\cup\Omega_{\alpha\delta}^-} \frac{f}{\mu_\infty}\bar{v} \quad \forall v \in V_{\alpha\delta},$$

where,

$$a^\delta(u,v) = \int_{\Omega_{\alpha\delta}^+\cup\Omega_{\alpha\delta}^-}\left(\nabla u\cdot\nabla\bar{v} - \frac{\omega^2\rho_\infty}{\mu_\infty}u\bar{v}\right) + i\omega\int_{S_{R_e}} u\bar{v}$$
$$+\frac{1}{A_0^\alpha\delta}\int_0^{2\pi}[u][\bar{v}] + \delta B_0^\alpha\int_0^{2\pi}\langle u\rangle_\alpha\langle\bar{v}\rangle_\alpha - \delta B_2^\alpha\int_0^{2\pi}\langle\frac{\partial u}{\partial\theta}\rangle_\alpha\langle\frac{\partial\bar{v}}{\partial\theta}\rangle_\alpha,$$

it is easily seen that $a^\delta$ can be split into a compact part and and a coercive one.
We can now prove that problem (64) is well-posed and is stable uniformly with respect to $\delta$.

**Proposition 3.4.** *Assume that $\alpha$ is such that $A_0^\alpha > 0$ and $B_2^\alpha < 0$. Then, for any $\omega > 0$, there exist $C_\omega > 0$ and $\delta_0 > 0$ such that, for any $\delta < \delta_0$, for any $u \in V_{\alpha\delta}$.*

$$\|u\|_{V_{\alpha\delta}} \leq C_\omega \sup_{v \in V_{\alpha\delta}, v\neq 0} \frac{a^\delta(u,v)}{\|v\|_{V_{\alpha\delta}}}. \tag{65}$$

*Proof.* The proof is done by contradiction: let $\delta_n$ be a sequence which tends to $0$ when $n$ tends to $+\infty$. We assume that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that:

$$\|u_n\|_{V_{\alpha\delta_n}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \sup_{v\in V^{\alpha\delta_n}, v\neq 0} \frac{a^{\delta_n}(u_n, v)}{\|v\|_{V_{\alpha\delta_n}}} = 0. \tag{66}$$

In order to work in a fix domain (independent of $\delta$), we shall consider $F^{\delta-}$ and $F^{\delta+}$:

$$F^{\delta-} := \begin{cases} [0,R_0] \to [0, R_0 - \alpha\delta], \\ \hat{x} \mapsto \frac{R_0-\alpha\delta}{R_0}\hat{x}, \end{cases} \qquad F^{\delta+} := \begin{cases} [R_0, R_e] \to [R_0+\alpha\delta, R_e], \\ \hat{x} \mapsto \left(\frac{R_e - R_0 - \alpha\delta}{R_e - R_0}\,|\hat{x}| + \frac{\alpha\delta R_e}{R_e - R_0}\right)\frac{\hat{x}}{|\hat{x}|}. \end{cases} \tag{67}$$

Since $DF^{\delta+}(\hat{x}) = \dfrac{R_e - R_0 - \alpha\delta}{R_e - R_0} I$ and $DF^{\delta-}(\hat{x}) = \dfrac{R_0 - \alpha\delta}{R_0} I$, $DF^{\pm}$ uniformly tends to the identity matrix when $\delta$ tends to $0$. The same is true of $\left|\det(DF^{\delta\pm})\right|$ which tends to $1$. Moreover there are $C_1^\pm$ and $C_2^\pm$ independent of $\delta$ such that

$$\begin{aligned} &C_1^\pm \left|\hat{h}\right| < \left|DF^{\delta\pm}\hat{h}\right| < C_2^\pm\left|\hat{h}\right| \quad \forall \hat{h}\in\mathbb{R}^2, \\ &C_1^\pm < \left|\det(DF^{\delta\pm})\right| < C_2^\pm. \end{aligned}$$

We also introduce three open sets independent of $\delta$,

$$\Omega^+ = B(0,R_e)\setminus \bar{B}(0,R_0), \qquad \Omega^- = B(0,R_0), \qquad \Omega = \Omega^+\cup\Omega^-,$$

and the Hilbert space $V_0$

$$V_0 = \left\{ v \in H^1(\Omega^+)\cap H^1(\Omega^-), \langle v\rangle_{|S_{R_0}} \in H^1_{per}(S_{R_0})\right\},$$

equipped with the norm

$$\|v\|_{V_0} = \|v\|^2_{H_1(\Omega^+)} + \|v\|^2_{H_1(\Omega^-)} + \frac{1}{\delta A_0^\alpha}\int_0^{2\pi} |[v(\theta)]_\alpha|^2\, d\theta - \delta B_2^\alpha \mu_\infty \int_0^{2\pi}\left( |\langle v\rangle|^2 + \left|\langle\frac{\partial v}{\partial\theta}\rangle\right|^2\right) d\theta. \tag{68}$$

Note that $F^{\pm\delta}$ transforms $\Omega^\pm_{\alpha\delta}$ into $\Omega^\pm$. Then we define $\hat{u}_n$

$$\hat{u}_n(\hat{x}) := \begin{cases} u_n \circ F^{\delta+}(\hat{x}) & \text{if } |\hat{x}| < R_0, \\ u_n\circ F^{\delta-}(\hat{x}) & \text{if } |\hat{x}| > R_0, \end{cases}$$

and the bilinear form $\hat{a}^{\delta_n}$

$$\begin{aligned}
\hat{a}^{\delta_n}(\hat{u}_n,\hat{v}) &:= a^{\delta_n}(u_n,v),\\
&:= \int_{\Omega^+}(DF^{\delta+}(\hat{x})^{-1})(DF^{\delta+}(\hat{x})^{-1*})\mu_\infty\hat{\nabla}\hat{u}_n\cdot\overline{\hat{\nabla}\hat{v}}\ \left|\det(DF^{\delta+})\right| + \int_{S_{R_e}} i\omega\mu_\infty\hat{u}_n\bar{v}d\sigma\\
&+\int_{\Omega^-}(DF^{\delta-}(\hat{x})^{-1})(DF^{\delta-}(\hat{x})^{-1*})\mu_\infty\hat{\nabla}\hat{u}_n\cdot\overline{\hat{\nabla}\hat{v}}\ \left|\det(DF^{\delta-})\right|\\
&-\int_{\Omega^+}\rho_\infty\omega^2\hat{u}_n\bar{\hat{v}}\ \left|\det(DF^{\delta+})\right| - \int_{\Omega^-}\rho_\infty\omega^2\hat{u}_n\bar{\hat{v}}\ \left|\det(DF^{\delta-})\right|\\
&-\delta B_2^\alpha\mu_\infty\int_0^{2\pi}\langle\frac{\partial\hat{u}_n}{\partial\theta}\rangle\langle\frac{\partial\bar{\hat{v}}}{\partial\theta}\rangle d\theta + \delta B_1^\alpha\mu_\infty\int_0^{2\pi}\langle\hat{u}_n\rangle\langle\hat{v}\rangle d\theta\\
&+\frac{\mu_\infty}{\delta A_0^\alpha}\int_0^{2\pi}[\hat{u}_n][\bar{\hat{v}}]d\theta \quad \forall\hat{v}\in V^0.
\end{aligned}\tag{69}$$

Using the properties of $F^{\delta\pm}$ and (66), we can assert that there exist two constants $A$ and $B$ independent of $\delta$ such that

- $$0 < A \leq \|\hat{u}_n\|_{V_0} \leq B, \tag{70}$$
- $$\sup_{\hat{v}\in V^0,\hat{v}\neq 0}\frac{\hat{a}^{\delta_n}(\hat{u}_n,\hat{v})}{\|\hat{v}\|_{V_0}} = 0. \tag{71}$$

Therefore, there is a sub-sequence (still denoted by $(\hat{u}_n)$) and a function $\hat{u}_0 \in H^1(\Omega^+)\cup H^1(\Omega^-)$ such that

$$\begin{aligned}
&\hat{u}_n \rightharpoonup \hat{u}_0^+ \text{ weakly in } H^1(\Omega^+),\\
&\hat{u}_n \rightharpoonup \hat{u}_0^- \text{ weakly in } H^1(\Omega^-),\\
&\hat{u}_n \rightharpoonup \hat{u}_0^\pm \text{ weakly in } H^{1/2}(S_{R_0^\pm}).
\end{aligned}$$

In addition, it follows from (70) that $\frac{1}{\delta A_0^\alpha}\int_0^{2\pi}\left|[\hat{u}_n]\right|^2 d\theta \leq B$. Consequently, by uniqueness of the weak limit,

$$\lim_{n\to\infty}\int_0^{2\pi}\left|[\hat{u}_n]\right|^2 d\theta = \int_0^{2\pi}\left|[\hat{u}_0]\right|^2 d\theta = 0.$$

So $[\hat{u}_0]_{S_{R_0}} = 0$ and $\hat{u}_0$ is in $H^1(\Omega)$.
Moreover, since $\|\sqrt{\delta}\langle\hat{u}_n\rangle\|_{H_1(]0,2\pi[)}$ is bounded, we also have

$$\lim_{n\to\infty} -\delta B_2^\alpha\mu_\infty\int_0^{2\pi}\langle\frac{\partial\hat{u}_n}{\partial\theta}\rangle\langle\frac{\partial\bar{v}}{\partial\theta}\rangle d\theta + \delta B_1^\alpha\mu_\infty\int_0^{2\pi}\langle\hat{u}_n^-\rangle\langle\bar{v}^-\rangle d\theta = 0 \qquad \forall v\in V_0.$$

Therefore, letting $n$ tends to $+\infty$ in the bilinear form (69) and using test functions in $H^1(\Omega)\cap V_0$, yields

$$0 = \int_\Omega \mu_\infty\nabla\hat{u}_0\cdot\nabla\bar{v} - \omega^2\rho_\infty\hat{u}_0\bar{v} + \int_{S_{R_e}} i\mu_\infty\omega\hat{u}_0\bar{v} \quad \forall v\in H^1(\Omega)\cap V_0.$$

By density of $H^1(\Omega)\cap V_0$ in $H^1(\Omega)$, the previous equality also holds for any $v\in H^1(\Omega)$:

$$0 = \int_\Omega \mu_\infty\nabla\hat{u}_0\cdot\nabla\bar{v} - \omega^2\rho_\infty\hat{u}_0\bar{v} + \int_{S_{R_e}} i\mu_\infty\omega\hat{u}_0\bar{v} \quad \forall v\in H^1(\Omega).$$

The previous formulation is the classical variational formulation associated to the homogeneous Helmholtz equation with first order approximation of the Sommerfeld radiation condition: hence $\hat{u}_0 = 0$. So $\hat{u}_n$ strongly tends to $0$ in $L^2(\Omega^{\pm})$ and $\sqrt{\delta}\hat{u}_n^-$ strongly tends to $0$ in $L^2(]0, 2\pi[)$. To obtain a contradiction we only need to check that $\|\hat{u}_n\|^2_{V^0}$ tends to $0$. But,this verified since the right side hand of the following inequality tends to $0$.

$$\|\hat{u}_n\|^2_{V^0} \leq C\left(\left|\hat{a}^\delta(\hat{u}_n, \hat{u}_n)\right| + \|\hat{u}_n\|^2_{L^2(\Omega^+)} + \|\hat{u}_n\|^2_{L^2(\Omega^-)} + \|\sqrt{\delta}\langle\hat{u}_n\rangle\|^2_{L^2(]0,2\pi[)}\right)$$

□

A natural question now is to ask if $\delta_0$ depends on $\omega$. The answer is unfortunately positive:

**Proposition 3.5.** *Let* $\tilde{B}_0^\alpha = \dfrac{B_0^\alpha}{\omega^2}$.

- *The set of frequencies such that* (64) *is ill-posed is included in* $B = \left\{k^2\dfrac{B_2^\alpha}{\tilde{B}_0^\alpha} + \dfrac{4}{\delta^2 A^\alpha}, \quad k \in \mathbb{N}\right\}$
- *Moreover, for $\delta$ small enough,*(64) *has non trivial solutions.*

*Proof.*

- Let us suppose that $u$ verifies (64) with $f = 0$. Combining first order radiation condition and Rellich lemma yields

$$u = 0 \text{ in } \Omega_\alpha^+. \tag{72}$$

Moreover, we also know that $u$ can be written as a Fourier Series in $\Omega_\alpha^-$

$$u := \sum_{n\in\mathbb{Z}} u_n(r)e^{in\theta} \tag{73}$$

Combining (72), (73) and the transmission conditions ((64)(a)-(b)) gives

$$\forall n \in \mathbb{Z}, \quad \begin{bmatrix} 1 & \dfrac{\delta A^\alpha}{2} \\ \delta(-n^2\dfrac{B_2^\alpha}{2} + \dfrac{\tilde{B}_0^\alpha\omega^2}{2}) & 1 \end{bmatrix} \begin{bmatrix} (u_n)_\alpha^- \\ \left(r\dfrac{\partial u_n}{\partial r}\right)_\alpha^- \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Consequently, $(u_n)_\alpha^-$ and $(r\dfrac{\partial u_n}{\partial r})_\alpha^-$ can be different from zero if the previous system is degenerated, which means that

$$\omega^2 = k^2\frac{B_2^\alpha}{\tilde{B}_0^\alpha} + \frac{4}{\delta^2 A_0^\alpha},$$

and the first part of the proposition is established.

We can also remark that if $\tilde{B}_0^\alpha < 0$, $B_2^\alpha < 0$ and $A_0^\alpha > 0$, for any $\omega > 0$, the problem is well-posed if $\delta^2 < \dfrac{4}{A_0^\alpha\omega^2}$.

- We now present a simple particular case where (64) has no-trivial solutions. We start from the exact problem (1-2) where

$$R_0 = 1, \qquad \mu^\delta = \begin{cases} \mu_0 < 1 & \text{if } 1 - \frac{\delta}{2} < r < 1 + \frac{\delta}{2} \\ 1 & \text{elsewhere,} \end{cases} \quad , \qquad \text{and } \rho^\delta = \mu^\delta.$$

In this particular configuration, the constants $A_0$, $B_0$ and $B_2$ are given by

$$A_0 = \frac{1-\mu_0}{\mu_0}, \qquad B_0 = \omega^2(1-\mu_0) \qquad \text{and } B_2 = (1-\mu_0).$$

To have $A_0^\alpha > 0$ and $B_0^\alpha < 0$, we can study (64) with $\alpha = 0.5$,

$$A_0^{1/2} = \frac{1}{\mu_0}, \qquad B_0^{1/2} = -\omega^2 \mu_0 \qquad \text{and} \quad B_2^{1/2} = -\mu_0.$$

Since $u$ is solution of an homogeneous Helmholtz equation in $\Omega^-$, it is clear that

$$u(r,\theta) = \sum_{n \in Z} c_n J_n(\omega r) e^{in\theta}.$$

Moreover, from the first part of the proof (72) we also know that $u$ is zero in $\Omega^+$. Consequently the transmission conditions $(64((a)-(b)))$ give

$$\forall n \in \mathbb{Z}, \quad \begin{cases} J_n'\left(\omega(1-\dfrac{\delta}{2})\right)\omega \dfrac{\delta A_0^{1/2}}{2} = -J_n\left(\omega(1-\dfrac{\delta}{2})\right), \\ -\omega J_n'\left(\omega(1-\dfrac{\delta}{2})\right) = \delta\left(\dfrac{\tilde{B}_0^{1/2}\omega^2}{2} - n^2\dfrac{B_2^{1/2}}{2}\right) J_n\left(\omega(1-\dfrac{\delta}{2})\right). \end{cases}$$

Hence, the problem has a no-zero solution if

$$\exists n \in \mathbb{Z}, \quad \begin{cases} \omega^2 = n^2 \dfrac{B_2^{1/2}}{\tilde{B}_0^{1/2}} + \dfrac{4}{\delta^2 A_0^{1/2}}, \\ J_n'\left(\omega(1-\dfrac{\delta}{2})\right) + J_n\left(\omega(1-\dfrac{\delta}{2})\right)\dfrac{2}{\omega\delta A_0^{1/2}} = 0, \end{cases} \quad (a) \quad \text{or} \quad \begin{cases} J_n\left(\omega(1-\dfrac{\delta}{2})\right) = 0, \\ J_n'\left(\omega(1-\dfrac{\delta}{2})\right) = 0, \end{cases} \quad (b)$$

For $n = 0$, there exists $\mu_0$ such that (a) holds. Actually, in this case,

$$\omega(\mu_0) = \frac{2}{\delta\sqrt{A_0^{1/2}(\mu_0)}} = \frac{2\sqrt{\mu_0}}{\delta}.$$

Using the fact that $J_0' = -J_1$ (see for instance [22]), we are interested in the zeros of the function $g(\mu_0)$ in the domain of validation of our model (i.e $\omega\delta \le 0.1 \Rightarrow \mu_0 < 0.025$),

$$g(\mu_0) = -J_1\left(\omega(\mu_0)(1-\frac{\delta}{2})\right) + \sqrt{\mu_0} J_0\left(\omega(\mu_0)(1-\frac{\delta}{2})\right).$$

As we can see on the fig.6, $g$ oscillates and has zeros.When $\delta$ tends to zero, it seems that $g$ has more and more zeros.

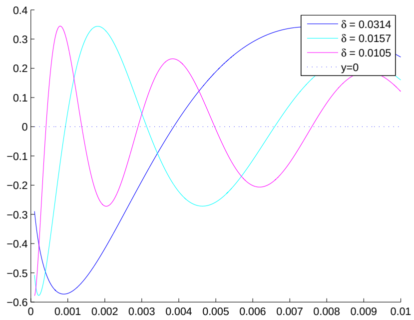



Figure 6: Graph of $g$ with respect to $\mu_0$

$\square$

### 3.1.3 Uncentered Approximate Problem

It is nevertheless possible to restore the uniqueness for any $(\delta, \omega) \in (\mathbb{R}_+^\star)^2$ using uncentered transmission conditions. Let $\breve{V}_{\alpha\delta} := H^1(\Omega_{\alpha\delta}^+) \cup H^1(\Omega_{\alpha\delta}^-) \cup H^1_{per}(S_{R_0-\alpha\delta})$ and its associated norm

$$\|\breve{v}\|^2_{\breve{V}_{\alpha\delta}} = \|\breve{v}\|^2_{H^1(\Omega^+_{\alpha\delta})} + \|\breve{v}\|^2_{H^1(\Omega^-_{\alpha\delta})} - \delta B_2^\alpha \|\breve{v}\|^2_{H^1(S_{R_0-\alpha\delta})} + \frac{1}{\delta A_0^\alpha} \int_0^{2\pi} |[\breve{v}(R_0 - \alpha\delta, \theta)]|^2 \, d\theta.$$

We introduce the following problem: find $\breve{u}_1^\delta \in \breve{V}_{\alpha\delta}$ such that

$$\begin{cases} \Delta \breve{u}_1^\delta + \omega^2 \dfrac{\rho_\infty}{\mu_\infty} \breve{u}_1^\delta = \dfrac{f}{\mu_\infty} \quad \text{in } \Omega^+_{\alpha\delta} \cup \Omega^-_{\alpha\delta}, \\ \left[\breve{u}_1^\delta\right]_\alpha = \delta A_0^\alpha (r \dfrac{\partial \breve{u}_1^\delta}{\partial r})^+_\alpha, \\ \left[r \dfrac{\partial \breve{u}_1^\delta}{\partial r}\right] = \delta \left( B_0^\alpha (\breve{u}_1^\delta)^-_\alpha + B_2^\alpha (\dfrac{\partial^2 \breve{u}_1^\delta}{\partial \theta^2})^-_\alpha \right), \\ \dfrac{\partial \breve{u}_1^\delta}{\partial r} + i\omega \breve{u}_1^\delta = 0 \quad \text{on } S_{R_e}, \end{cases} \tag{74}$$

We also consider its variational formulation: find $\breve{u}_1^\delta \in \breve{V}_{\alpha\delta}$ such that,

$$\breve{a}^\delta(\breve{u}_1^\delta, v) = -\int_{\Omega^+_{\alpha\delta} \cup \Omega^-_{\alpha\delta}} \frac{f}{\mu_\infty} \bar{v} \quad \forall \breve{v} \in \breve{V}_{\alpha\delta},$$

where,

$$\begin{aligned} \breve{a}^\delta(u, v) = & \int_{\Omega^+_{\alpha\delta} \cup \Omega^-_{\alpha\delta}} \left( \nabla u \cdot \nabla \bar{v} - \frac{\omega^2 \rho_\infty}{\mu_\infty} u \bar{v} \right) + i\omega \int_{S_{R_e}} u \bar{v} \\ & + \frac{1}{A_0^\alpha \delta} \int_0^{2\pi} [u][\bar{v}] + \delta B_0^\alpha \int_0^{2\pi} (u)^-_\alpha (\bar{v})^-_\alpha - \delta B_2^\alpha \int_0^{2\pi} \left( \frac{\partial u}{\partial \theta} \right)^-_\alpha \left( \frac{\partial \bar{v}}{\partial \theta} \right)^-_\alpha . \end{aligned} \tag{75}$$

Note that we have replaced the mean values by the exterior or interior values. Of course these modifications are consistent of order $\delta^2$.

**Proposition 3.6.** *. Assume that $\alpha$ is such that $A_0^\alpha > 0$ and $B_2^\alpha < 0$. Then, for any $\delta > 0$, for any $\omega > 0$, Problem* (74) *is well-posed. Moreover,*

$$\forall \delta_0 > 0,\ \forall\ \omega > 0,\ \exists\ C_\omega^{\delta_0} > 0,\ \forall \delta < \delta_0,\ \forall\ \breve{u} \in \breve{V}_{\alpha\delta} \quad \|\breve{u}\|_{\breve{V}_{\alpha\delta}} \leq C_\omega^{\delta_0} \sup_{\breve{v}\in\breve{V}_{\alpha\delta},\breve{v}\neq 0} \frac{\breve{a}^\delta(\breve{u},\breve{v})}{\|\breve{v}\|_{\breve{V}_{\alpha\delta}}}. \tag{76}$$

*Proof.*

- Let us fix $\delta$ and $\omega$. It easily seen that (74) has a unique solution. It follows from the application of the Rellich Lemma and the use of the uncentered transmission conditions. (Indeed, if $(u)_\alpha^+ = \left(r\dfrac{\partial u}{\partial r}\right)_\alpha^+ = 0$, a trivial verification shows that $(u)_\alpha^- = \left(r\dfrac{\partial u}{\partial r}\right)_\alpha^- = 0$). Consequently,

$$\forall \delta > 0, \forall \omega > 0, \exists\ C_\omega^\delta > 0 \text{ such that } \|\breve{u}\|_{V_{\alpha\delta}} \leq C_\omega^\delta \sup_{\breve{v}\in\breve{V}_{\alpha\delta},\breve{v}\neq 0} \frac{\breve{a}^\delta(\breve{u},\breve{v})}{\|\breve{v}\|_{\breve{V}_{\alpha\delta}}} \tag{77}$$

- Moreover, the same method as in the proof of (65) applies and yields

$$\forall\ \omega > 0,\ \exists\ C_\omega > 0,\ \exists\ \delta_0(\omega) > 0,\ \forall\ \delta < \delta_0, \quad \|\breve{u}\|_{V_{\alpha\delta}} \leq C_\omega \sup_{\breve{v}\in\breve{V}_{\alpha\delta},\breve{v}\neq 0} \frac{\breve{a}^\delta(\breve{u},\breve{v})}{\|\breve{v}\|_{\breve{V}_{\alpha\delta}}}. \tag{78}$$

- It suffices now to prove that $\delta_0$ is independent of $\omega$ ( $\inf_{\omega\in\mathbb{R}_+^\star} \delta_0(\omega) > 0$): the proof is done by contradiction. The contradiction of (76) implies that there are $\delta_0 > 0$ and $\omega > 0$ such that,

$$\forall n \in \mathbb{N}, \exists\ \delta_n \text{ such that } 0 < \delta_n < \delta_0, \exists\ \breve{u}_{\delta_n} \quad \|\breve{u}^{\delta_n}\|_{V^{\delta_n}} >\ n \sup_{\breve{v}\in\breve{V}_{\alpha\delta^n},\breve{v}\neq 0} \frac{\breve{a}^{\delta_n}(\breve{u}^{\delta_n},\breve{v})}{\|\breve{v}\|_{\breve{V}_{\alpha\delta}}}.$$

  By (77), $\forall n \in \mathbb{N},\ C_\omega^{\delta_n} > n$. Then $\lim_{n\to\infty} C_\omega^{\delta_n} = +\infty$.
  But, by (78), $\exists n_0 \in \mathbb{N},\ \forall n \geq n_0, \delta(\omega) < \delta_n < \delta_0$. Consequently $\delta_n$ is bounded, and up to a subsequence $\delta_n$ is going to $\delta_\star$ when $n$ is going to $+\infty$. This implies that $C_\omega^{\delta_\star} \geq +\infty$, which contradicts (77).

□

### 3.1.4 Convergence Estimates

The proof of convergence is done for the uncentered problem (74). the same method applies for the centered problem (64).

The classical method to prove a convergence result is divided in two main steps(see for example [16]) :

- The first step consists in doing an asymptotic expansion of the approximate solution $\breve{u}_1^\delta$ according to $\delta$. In this simple case, we do not need to do matched asymptotic expansion: the classical expansion $\breve{u}_1^\delta = \sum_{n\in\mathbb{N}} \delta^n \breve{v}_n$ is valid in the whole domain $\Omega^+ \cup \Omega^-$. We also can easily deduce optimal error estimate using the stability result (65). Moreover, the two first terms $\breve{v}_0$ and $\breve{v}_1$ coincide with the two first term $u_0$ and $u_1$ of the far field expansion of the exact solution.

- Then, the convergence estimate follows from a triangular inequality

$$\|u^\delta - \breve{u}_1^\delta\|_{H_1(\Omega_\gamma^\pm)} \leq \|u^\delta - (u_0 + \delta u_1)\|_{H_1(\Omega_\gamma^\pm)} + \|\breve{u}_1^\delta - (u_0 + \delta u_1)\|_{H_1(\Omega_\gamma^\pm)}.$$

Assume that $\breve{u}_1^\delta = \sum_{n\in\mathbb{N}} \delta^n \breve{v}_n$. Then, the fields $\breve{v}_n$ are uniquely defined by induction and are solution of the following problems: find $u_n \in H^1(\Omega^+)\cap H^1(\Omega^-)$ such that,

$$\begin{cases} \Delta \breve{v}_n + \dfrac{\omega^2\rho_\infty}{\mu_\infty}\breve{v}_n = \delta_0(n)\dfrac{f}{\mu_\infty} \text{ in } \Omega^+\cup\Omega^-, \\ [\breve{v}_n] = A_0(r\dfrac{\partial \breve{v}_n}{\partial r})^+ + \displaystyle\sum_{l=1}^{n-1}\frac{\alpha^l}{(l-1)!}A_0\left(\frac{R_0}{l}(\frac{\partial^{l+1}\breve{v}_{n-1-l}}{\partial r^{l+1}})^+ + (\frac{\partial^l \breve{v}_{n-1-l}}{\partial r^l})^+\right) \\ \qquad -\displaystyle\sum_{l=1}^{n}\frac{\alpha^l}{l!}\left((\frac{\partial^l\breve{v}_{n-l}}{\partial r^l})^+ - (-1)^l(\frac{\partial^l\breve{v}_{n-l}}{\partial r^l})^-\right) \text{ on } S_{R_0}, \\ \left[r\dfrac{\partial\breve{v}_n}{\partial r}\right] = \displaystyle\sum_{l=0}^{n-1}\frac{(-\alpha)^l}{l!}\left(B_0(\frac{\partial^l\breve{v}_{n-1-l}}{\partial r^l})^- + B_2(\frac{\partial^{l+2}\breve{v}_{n-1-l}}{\partial r^l\partial\theta^2})^-\right) \\ \qquad -\displaystyle\sum_{l=1}^{n}\frac{(\alpha)^l}{(l-1)!}\left(\frac{R_0}{l}\left((\frac{\partial^{l+1}\breve{v}_{n-l}}{\partial r^{l+1}})^+ - (-1)^l(\frac{\partial^{l+1}\breve{v}_{n-l}}{\partial r^{l+1}})^-\right) + \left((\frac{\partial^l\breve{v}_{n-l}}{\partial r^l})^+ - (-1)^l(\frac{\partial^l\breve{v}_{n-l}}{\partial r^l})^-\right)\right). \end{cases}$$

**Remark 3.7.** *Note that $\breve{v}_0 = u_0$ and $\breve{v}_1 = u_1$.*

We have the following consistency result

**Proposition 3.8.** *For any $n\in\mathbb{N}$ , there is $C_n$ independent of $\delta$ such that*

$$\|\breve{u}_1^\delta - \sum_{k=1}^n \delta^k\breve{v}_k\|_{H_1(\Omega^\pm_{\alpha\delta})} \le C_n\delta^{n+1}. \tag{79}$$

*Proof.* Set $\varepsilon_n = \breve{u}_1^\delta - \sum_{k=1}^n \delta^k\breve{v}_k$. $\varepsilon_n$ is the solution of the following problem: find $\varepsilon_n\in H^1(\Omega^+)\cap H^1(\Omega^-)$ such that

$$\begin{cases} -\mu_\infty\Delta\varepsilon_n - \rho_\infty\omega^2\varepsilon_n = 0, \\ \dfrac{\partial\varepsilon_n}{\partial r} + i\omega\varepsilon_n = 0 \text{ on } S_{R_e}, \\ [\varepsilon_n]_\alpha = \delta A^\alpha(r\dfrac{\partial\varepsilon_n}{\partial r})^+_\alpha + g_n(\theta), \\ \left[r\dfrac{\partial\varepsilon_n}{\partial r}\right]_\alpha = \delta B_1^\alpha\varepsilon_n^- + \delta B_2^\alpha\dfrac{\partial^2\varepsilon_n^-}{\partial\theta^2} + h_n(\theta), \end{cases}$$

where $g_n$ and $h_n$ are in $L^\infty(]0,2\pi[)$ satisfying the following estimates

$$\|h_n\|_{L^\infty(]0,2\pi[} \le C_n\delta^{n+1} \quad \text{and} \quad \|g_n\|_{L^\infty(]0,2\pi[} \le C_n\delta^{n+1}.$$

Consequently,

$$\begin{aligned} a^\delta(\varepsilon_n,\breve{v}) &= \frac{1}{\delta A^\alpha}\int_0^{2\pi} g_n[\bar{\breve{v}}]d\theta + \int_0^{2\pi} h_n\bar{\breve{v}}^-, \\ &\le \frac{C}{\sqrt{\delta}}\left(\frac{1}{\delta A^\alpha}\int_0^{2\pi}\left|[\bar{\breve{v}}]\right|^2 d\theta\right)^{1/2}\|g_n\|_{L^\infty[0,2\pi[} + C\|h_n\|_{L^\infty[0,2\pi[}\|\breve{v}\|_{H^1(S_{R_0-\alpha\delta})}, \\ &\le C\delta^{n+1/2}\|\breve{v}\|_{\breve{V}_{\alpha\delta}}. \end{aligned}$$

Combining the previous inequality with the stability result of the approximate problem (65), we can assert that

$$\|\varepsilon_n\|_{H^1(\Omega^\pm_\delta)} \le \|\varepsilon_n\|_{V_{\alpha\delta}} \le C\delta^{n+1/2}$$

Using the triangular inequality gives an optimal error estimate and completes the proof. □

We can now state our main result:

**Proposition 3.9.** *We recall that $\Omega_\gamma$ is defined by* (60) *and that $u^\delta$ is the solution of the exact problem (1-2). For any $\gamma > 0$, there exists $\delta_0$ such that, for $\delta < \delta_0$*

$$\|u^\delta - \breve{u}_1^\delta\|_{H^1(\Omega_\gamma)} \leq C\delta^2.$$

*Proof.* The proof is immediate. It follows from the triangular inequality and the results (61) and (79).

$$\|u^\delta - \breve{u}_1^\delta\|_{H^1(\Omega_\gamma)} \leq \|u^\delta - u_0 - \delta u_1\|_{H^1(\Omega_\gamma)} + \|u_0 + \delta u_1 - \breve{u}_1^\delta\|_{H^1(\Omega_\gamma)} \leq C\delta^2.$$

□

## 3.2 Second Order Approximate Problem in the Symmetric Case

The goal of this section is to find an approximate solution of order $\delta^2$: we look for a problem whose the solution is close to $u_0 + \delta u_1 + \delta^2 u_2$. We begin by reminding the transmission conditions of $u_0$, $u_1$ and $u_2$ across the interface $S_{R_0}$:

$$\begin{aligned}
&[u_0] = 0, && \left[r\frac{\partial u_0}{\partial r}\right] = 0,\\
&[u_1] = A_0\ \langle r\frac{\partial u_0}{\partial r}\rangle, && \left[r\frac{\partial u_1}{\partial r}\right] = B_0\ \langle u_0\rangle\ +\ B_2\ \langle\frac{\partial^2 u_0}{\partial \theta^2}\rangle,\\
&[u_2] = A_0\ \langle r\frac{\partial u_1}{\partial r}\rangle, && \left[r\frac{\partial u_2}{\partial r}\right] = B_0\ \langle u_1\rangle\ +\ B_2\langle\frac{\partial^2 u_1}{\partial \theta^2}\rangle.
\end{aligned}$$

It is clear that we encounter the same difficulties as in the first order approximate conditions. Consequently, we choose to use the same approach as in the previous section to obtain a well-posed problem: we shift the jump from $\alpha\delta$ and we use uncentered transmission conditions. Set $\tilde{u}_2^\delta = u_0 + \delta u_1 + \delta^2 u_2$. An easy computation yields

$$\begin{cases}
\left[\tilde{u}_2^\delta\right]_\alpha = \delta A_0^\alpha \left(r\dfrac{\partial \tilde{u}_2^\delta}{\partial r}\right)_\alpha^+ + \delta^2 A_1^\alpha \left(\dfrac{\partial^2 \tilde{u}_2^\delta}{\partial \theta^2}\right)_\alpha^\pm + \delta^2 A_2^\alpha \left(\tilde{u}_2^\delta\right)_\alpha^\pm,\\[2ex]
\left[r\dfrac{\partial \tilde{u}_2^\delta}{\partial r}\right]_\alpha = \delta B_0^\alpha \left(\tilde{u}_2^\delta\right)_\alpha^- + \delta B_2 \left(\dfrac{\partial^2 \tilde{u}_2^\delta}{\partial \theta^2}\right)_\alpha^- - \delta^2 A_2^\alpha \left(r\dfrac{\partial \tilde{u}_2^\delta}{\partial r}\right)_\alpha^\pm - \delta^2 A_1^\alpha \dfrac{\partial^2}{\partial \theta^2}\left(r\dfrac{\partial \tilde{u}_2^\delta}{\partial r}\right)_\alpha^\pm,
\end{cases}$$

where,

$$A_0^\alpha = A_0 + \frac{2\alpha}{R_0}, \qquad A_1^\alpha = \alpha\left(\frac{A_0^\alpha}{R_0} - \frac{B_2}{R_0}\right), \quad A_2^\alpha = \alpha\left(\frac{\omega^2\rho_\infty R_0}{\mu_\infty}A_0^\alpha - \frac{B_0}{R_0}\right),$$

$$B_0^\alpha = B_0 - \frac{2\alpha R_0\rho_\infty\omega^2}{\mu_\infty}, \quad B_2^\alpha = B_2 - \frac{2\alpha}{R_0}.$$

In order to obtain a variational problem, we write the transmission conditions in a convenient form

$$\left[(\tilde{v}_2^\alpha)_\alpha^+ - ((1+\delta^2 A_2^\alpha)Id + \delta^2 A_1^\alpha\frac{\partial^2}{\partial\theta^2})(\tilde{u}_2^\delta)_\alpha^-\right] = \delta A_0^\alpha \left(r\frac{\partial \tilde{u}_2^\delta}{\partial r}\right)_\alpha^+,$$

$$\left[((1+\delta^2 A_2^\alpha)Id + \delta^2 A_1^\alpha\frac{\partial^2}{\partial\theta^2})\left(r\frac{\partial \tilde{u}_2^\delta}{\partial r}\right)_\alpha^+ - \left(r\frac{\partial \tilde{u}_2^\delta}{\partial r}\right)_\alpha^-\right] = \delta\left(B_0^\alpha(\tilde{u}_2^\delta)_\alpha^- + B_2^\alpha(\frac{\tilde{u}_2^\delta}{\partial\theta^2})_\alpha^-\right).$$

From now, we choose $\alpha$ such that $A_0^\alpha$, $A_1^\alpha$ and $B_0^\alpha$ are positive and $B_2^\alpha$ is negative. We first remark that $((1+\delta^2 A_2^\alpha)Id + \delta^2 A_1^\alpha \dfrac{\partial^2}{\partial\theta^2})$ is not always positive. Nevertheless,

$$((1+\delta^2 A_2^\alpha)Id + \delta^2 A_1^\alpha \frac{\partial^2}{\partial\theta^2}) = (1+\delta^2 A_2^\alpha)\left(Id - \delta^2 A_1^\alpha \frac{\partial^2}{\partial\theta^2}\right)^{-1} + O(\delta^3).$$

If we neglect the rest in $\delta^3$, we can replace $\left((1+\delta^2 A_2^\alpha)Id + \delta^2 A_1^\alpha \dfrac{\partial^2}{\partial\theta^2}\right)$ by the positive operator $(1+\delta^2 A_2^\alpha)\left(Id - \delta^2 A_1^\alpha \frac{\partial^2}{\partial\theta^2}\right)^{-1}$.

To shorten notation we introduce the operator $T^\delta$ and $\mathcal{T}^\delta$ :

$$T^\delta : \begin{cases} H^2_{per}(]0,2\pi[) \to L^2(]0,2\pi[), \\ u \mapsto (Id - \delta^2 A_1^\alpha \dfrac{\partial^2}{\partial\theta^2})u, \end{cases}$$

$$\mathcal{T}^\delta : \begin{cases} L^2_{per}(]0,2\pi[) \to H^2(]0,2\pi[), \\ u \mapsto (1+\delta^2 A_2^\alpha)T^{-1}(u). \end{cases}$$

Note that since $T$ is self-adjoint positive, $\mathcal{T}^\delta$ is also self-adjoint and positive.
Finally, we propose the following second order approximate problem: find $u_2^\delta \in V_{\alpha\delta}$, such that

$$\begin{cases} \Delta u_2^\delta + \dfrac{\omega^2\rho_\infty}{\mu_\infty} u_2^\delta = \dfrac{f}{\mu_\infty} \quad \text{in } \Omega^+_{\alpha\delta} \cup \Omega^-_{\alpha\delta}, \\ (u_2^\delta)^+_\alpha - \mathcal{T}^\delta (u_2^\delta)^-_\alpha = \delta A_0^\alpha \left(r\dfrac{\partial u_2^\delta}{\partial r}\right)^+_\alpha, \\ \mathcal{T}^\delta\,[\,\left(r\dfrac{\partial u_2^\delta}{\partial r}\right)^+_\alpha\,] - \left(r\dfrac{\partial u_2^\delta}{\partial r}\right)^-_\alpha = \delta\left(B_0^\alpha (u_2^\delta)^-_\alpha + B_2^\alpha (\dfrac{\partial^2 u_2^\delta}{\partial\theta^2})^-_\alpha\right), \\ \dfrac{\partial u_2^\delta}{\partial r} + i\omega u_2^\delta = 0, \quad \text{on } S_{R_e}. \end{cases} \tag{80}$$

and its associated variational formulation: find $u_2^\delta \in V_{\alpha\delta}$,

$$a_2^\delta(u_2^\delta, v) = \int_{\Omega_{\alpha\delta}} \frac{f}{\mu_\infty}\bar{v} \quad \forall\, v \;\in\; V_{\alpha\delta}, \tag{81}$$

where,

$$a_2^\delta(u,v) = \int_{\Omega_{\alpha\delta}} \nabla u\cdot\nabla\bar{v} - \frac{\omega^2\rho_\infty}{\mu_\infty}u\bar{v} + \int_{S_{R_e}} i\omega u\bar{v} - \delta B_2^\alpha \int_0^{2\pi} (\frac{\partial u}{\partial\theta})^-_\alpha (\frac{\partial \bar{v}}{\partial\theta})^-_\alpha$$
$$+ \frac{1}{A_0^\alpha\delta}\int_0^{2\pi}((u)^+_\alpha - \mathcal{T}^\delta\left[(u)^-_\alpha\right])((\bar{v})^+_\alpha - \mathcal{T}^\delta\left[(\bar{v})^-_\alpha\right]) + \delta B_0^\alpha\int_0^{2\pi}(u)^-_\alpha(\bar{v})^-_\alpha.$$

**Proposition 3.10.** *Let us equip the space $V_{\alpha\delta}$ with the norm*

$$\|u\|^2_{V_{\alpha\delta,2}} = \|u\|^2_{H^1(\Omega^+_{\alpha\delta})} + \|u\|^2_{H^1(\Omega^-_{\alpha\delta})} + \delta|B_2^\alpha|\;\|u\|^2_{H^1(]0,2\pi[)} + \frac{1}{\delta A_0^\alpha}\int_0^{2\pi}\left|\left((u)^+_\alpha - \mathcal{T}^\delta\left[(u)^-_\alpha\right]\right)\right|^2.$$

*For any frequency $\omega>0$ and for any $\delta>0$, the problem* (80) *is well-posed :*

$$\forall\delta_0>0,\;\forall\,\omega>0,\;\exists\,C_\omega^{\delta_0}>0,\;\forall\delta<\delta_0,\;\forall\,u\in V_{\alpha\delta}\quad \|u\|_{V_{\alpha\delta},2} \le C_\omega^{\delta_0} \sup_{v\in V_{\alpha\delta}, v\neq 0}\frac{a_2^\delta(u,v)}{\|v\|_{V_{\alpha\delta},2}}. \tag{82}$$

*Moreover, for any* $\gamma > 0$,

$$\|u^\delta - u_2^\delta\|_{H^1(\Omega_\gamma)} \leq C\delta^3.$$

*Proof.* The proof is by contradiction. The main steps of the proof are the same as in the proof of the proposition 3.4 but small technical alterations are needed.

The proof start as in the proposition 3.4. Let $\delta_n$ be a sequence that tends to $0$ when $n$ tends to $+\infty$. We assume that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that

- $\|u_n\|_{V_{\alpha\delta_n,2}} = 1,$
- $\lim\limits_{n\to\infty} \sup\limits_{v\in V_{\alpha\delta_n}, v\neq 0} \dfrac{a^{\delta_n}(u_n, v)}{\|v\|_{V_{\alpha\delta_n,2}}} = 0.$

Using the transformations $F^{\delta^\pm}$, and defining $\hat{u}_n$ and $\hat{a}_2^{\delta_n}$ as in 3.4, we obtain

$$\bullet\ 0 < A \leq \|\hat{u}_n\|_{V_{0,2}} \leq B, \tag{83}$$

$$\bullet\ \sup_{\hat{v}\in V_0, \hat{v}\neq 0} \frac{\hat{a}^{\delta_n}(\hat{u}_n, \hat{v})}{\|\hat{v}\|_{V_{0,2}}} = 0. \tag{84}$$

where

$$\|\hat{u}_n\|_{V_{0,2}} = \|\hat{u}\|^2_{H^1(\Omega^+)} + \|\hat{u}\|^2_{H^1(\Omega^-)} + \delta_n\,|B_2^\alpha|\,\|(\hat{u})^-\|^2_{H^1(]0,2\pi[)} + \frac{1}{\delta_n A_0^\alpha}\int_0^{2\pi} \left|\left((\hat{u})^+ - \mathcal{T}_n^\delta\left[(\hat{u})^-\right]\right)\right|^2.$$

Therefore, as in 3.4, there is a sub-sequence (still denoted by $(\hat{u}_n)$) and a function $\hat{u}_0 \in H^1(\Omega^+)\cup H^1(\Omega^-)$ such that

$$\begin{aligned}
&\hat{u}_n \rightharpoonup \hat{u}_0^+ \text{ weakly in } H^1(\Omega^+),\\
&\hat{u}_n \rightharpoonup \hat{u}_0^- \text{ weakly in } H^1(\Omega^-),\\
&\hat{u}_n \rightharpoonup \hat{u}_0^\pm \text{ weakly in } H^{1/2}(S_{Rn_0^\pm}).
\end{aligned}$$

Now, as in 3.4, we prove that $[\hat{u}_0] = 0$. However, it is less immediate than in the previous cases.Using (83), it is clear that

$$\lim_{n\to\infty}\int_0^{2\pi} \left|\left((\hat{u}_n)^+ - \mathcal{T}^{\delta_n}\left[(\hat{u}_n)^-\right]\right)\right|^2 = 0.$$

Set $\hat{w}_n = \mathcal{T}^{\delta_n}\left[(\hat{u}_n)^-\right]$.By definition, $\hat{w}_n$ satisfies :

$$A_1^\alpha\delta_n^2\int_0^{2\pi}\frac{\partial\hat{w}_n}{\partial\theta}\frac{\partial\overline{w}}{\partial\theta} + \int_0^{2\pi}\hat{w}_n\overline{w} = \int_0^{2\pi}(1+\delta_n^2A_2^\alpha)(\hat{u}_n)^-\overline{w}\quad \forall w\in H^1_{per}(]0,2\pi[).$$

Consequently, choosing $w = \hat{w}_n - (1+\delta_n^2A_2^\alpha)(\hat{u}_n)^-$ yields

$$A_1^\alpha\delta_n^2\int_0^{2\pi}\left(\frac{\partial\hat{w}_n}{\partial\theta} - (1+\delta_n^2A_2^\alpha)\frac{\partial(\hat{u}_n)^-}{\partial\theta}\right)^2 + \int_0^{2\pi}\left(\hat{w}_n - (1+\delta_n^2A_2^\alpha)(\hat{u}_n)^-\right)^2 =$$
$$-\,A_1^\alpha\delta_n^2(1+\delta_n^2A_2^\alpha)\int_0^{2\pi}\left(\frac{\partial(\overline{\hat{u}}_n)^-}{\partial\theta}\right)\left(\frac{\partial\overline{\hat{w}}_n}{\partial\theta} - (1+\delta_n^2A_2^\alpha)\frac{\partial(\overline{\hat{u}}_n)^-}{\partial\theta}\right).$$

Therefore we deduce that

$$\|\frac{\partial \hat{w}_n}{\partial \theta} - (1+\delta_n^2 A_2^\alpha)\frac{\partial (\hat{u}_n)^-}{\partial \theta}\|_{L^2(]0,2\pi[)} \leq (1+\delta_n^2 A_2^\alpha)\|\frac{\partial (\hat{u}_n)^-}{\partial \theta}\|_{L^2(]0,2\pi[)},$$
$$\leq C\|\frac{\partial (\hat{u}_n)^-}{\partial \theta}\|_{L^2(]0,2\pi[)}.$$

and

$$\|\hat{w}_n - (1+\delta_n^2 A_2^\alpha)(\hat{u}_n)^-\|_{L^2(]0,2\pi[)}^2 \leq C^2 A_1^\alpha \delta_n^2 \|\frac{\partial (\hat{u}_n)^-}{\partial \theta}\|_{L^2(]0,2\pi[)}^2. \tag{85}$$

Moreover, since by assumption $\delta_n\|\frac{\partial (\hat{u}_n)^-}{\partial \theta}\|_{L^2(]0,2\pi[)}^2$ is bounded,we have

$$\lim_{n\to+\infty} \|\hat{w}_n - (1+\delta_n^2 A_2^\alpha)(\hat{u}_n)^-\|_{L^2(]0,2\pi[)} = 0.$$

Using the triangular inequality yields

$$\|\hat{w}_n - (\hat{u}_n)^-\|_{L^2(]0,2\pi[)} \leq \underbrace{\|\hat{w}_n - (1+\delta_n^2 A_2^\alpha)(\hat{u}_n)^-}_{\to 0}\|_{L^2(]0,2\pi[)} + \underbrace{\delta_n^2 A_2^\alpha \|(\hat{u}_n)^-\|_{L^2(]0,2\pi[)}}_{\to 0},$$

and

$$\lim_{n\to\infty} \|\mathcal{T}^{\delta_n}\left[(\hat{u}_n)^-\right] - (\hat{u}_n)^-\|_{L^2(]0,2\pi[)} = 0.$$

Using again the triangular inequality gives the desired result. Indeed, since,

$$\|(\hat{u}_n)^+ - (\hat{u}_n)^-\|_{L^2(]0,2\pi[)} \leq \underbrace{\|(\hat{u}_n)^+ - \mathcal{T}^{\delta_n}\left[(\hat{u}_n)^-\right]\|_{L^2(]0,2\pi[)}}_{\to 0} + \underbrace{\|\mathcal{T}^{\delta_n}\left[(\hat{u}_n)^-\right] - (\hat{u}_n)^-\|_{L^2(]0,2\pi[)}}_{\to 0},$$

it follows that

$$\lim_{n\to+\infty} \|(\hat{u}_n)^+ - (\hat{u}_n)^-\|_{L^2(]0,2\pi[)} = 0.$$

By uniqueness of the weak limit,

$$\lim_{n\to+\infty} \|(\hat{u}_n)^+ - (\hat{u}_n)^-\|_{L^2(]0,2\pi[)} = \|(\hat{u}_0)^+ - (\hat{u}_0)^-\|_{L^2(]0,2\pi[)} = 0.$$

Therefore,

$$[\hat{u}_0] = 0.$$

Let $\hat{v} \in H^1(\Omega)$ such that $(\hat{v})^- \in H^1_{per}(]0,2\pi[)$. As in (85), we can prove that

$$\frac{1}{\delta_n^2}\int_0^{2\pi} \left|\mathcal{T}^{\delta_n}\left[(\hat{v})^-\right] - (1+\delta_n^2 A_2^\alpha)(v)^-\right|^2 \leq C.$$

Consequently, by triangular inequality

$$\begin{aligned}\frac{1}{\delta_n^2}\int_0^{2\pi} \left|\mathcal{T}^{\delta_n}\left[(\hat{v})^-\right] - (\hat{v})^+\right|^2 &\leq \frac{C}{\delta_n^2}\left(\int_0^{2\pi} \left|\mathcal{T}^{\delta_n}\left[(\hat{v})^-\right] - (1+\delta_n^2 A_2^\alpha)(v)^-\right|^2 + \delta_n^2\|v\|_{L^2]0,2\pi[}^2\right),\\ &\leq C.\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \frac{1}{A_0^\alpha \delta_n} \int_0^{2\pi} \left( (\hat{u}_n)^+ - \mathcal{T}^{\delta_n} \left[ (\hat{u}_n)^- \right] \right) \left( \hat{v} - \mathcal{T}^{\delta_n} \left[ (\hat{v})^- \right] \right) \right| \\
&\qquad \leq C \| (\hat{u}_n)^+ - \mathcal{T}^{\delta_n} \left[ (\hat{u}_n)^- \right] \|_{L^2(]0,2\pi[)} \left( \frac{1}{\delta_n^2} \int_0^{2\pi} \left| \mathcal{T}^{\delta_n} \left[ (\hat{v})^- \right] - (\hat{v})^+ \right|^2 \right)^{\frac{1}{2}}, \\
&\qquad \leq C \underbrace{\| (\hat{u}_n)^+ - \mathcal{T}^{\delta_n} \left[ (\hat{u}_n)^- \right] \|_{L^2(]0,2\pi[)}}_{\to 0}.
\end{aligned}$$

Then, letting $n \to +\infty$ we can assert that,

$$\lim_{n \to +\infty} \frac{1}{A_0^\alpha} \int_0^{2\pi} \left( (\hat{u}_n)^+ - \mathcal{T}^{\delta_n} \left[ (\hat{u}_n)^- \right] \right) \left( \hat{v} - \mathcal{T}^{\delta_n} \left[ (\hat{v})^- \right] \right) = 0.$$

The rest of the proof exactly runs as in the proof of the proposition 3.4 : we prove that $u_0 = 0$. It follows that $\lim_{n \to \infty} \|\hat{u}_n\|_{V_{0,2}} = 0$ which contradicts the initial assumption. The proof of convergence is standard and is left to the reader. $\square$

# 4 Numerical Validation

## 4.1 Algorithm and Description of the Experiments

### 4.1.1 Algorithm

The numerical method is divided into three steps:

- Computation of $V_1^1$ and $W_0^0$ in order to determine $\alpha$, $A_0^\alpha$, $B_0^\alpha$ and $B_2^\alpha$,
- Solving the approximate problem with an unrefined computational mesh that ignores the presence of the periodic ring,

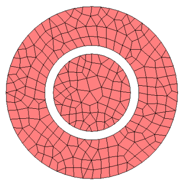


Figure 7: computational mesh

- Reconstruction of the solution in the periodic ring (optional).

### 4.1.2 Description of the Experiment

- The geometry of the periodicity cell is described on the figure 8. $\mu$ and $\rho$ are piecewise constants:

$$\left\{\begin{array}{lll} \mu_\infty = 1, & \mu_1 = 0.5, & \mu_2 = 2, \\ \rho_\infty = 1, & \rho_1 = 2, & \rho_2 = 4. \end{array}\right.$$

- The source is an incident plane wave.
- The frequency $\omega$ is equal to $2\pi$ and $R_0$, the mean radius of the periodic ring in equal to $1$.

We also define $N$, the number of periodic cells in the periodic ring,

$$N = \frac{2\pi R_0}{\delta}.$$

The numerical computations are done with the package Montjoie developed by M.Duruflé ([23]).

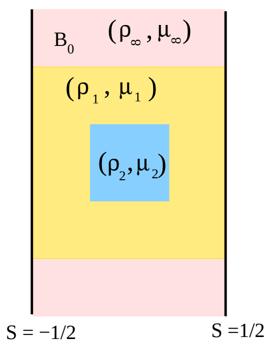



Figure 8: the periodicity cell

## 4.2 First Order Approximate Condition

We suppose that we have a sequence of finite-dimensional subspaces $(V^h_{\alpha\delta})$, $(h > 0)$ of the Hilbert space $\breve{V}_{\alpha\delta}$ such that:

**-H1** $\forall v \in \breve{V}_{\alpha\delta}, \exists (v_h)_{h>0}$ such that

$$\lim_{h\to 0}\Big(\|v - v_h\|_{H^1(\Omega^+_{\alpha\delta})} + \|v - v_h\|_{H^1(\Omega^-_{\alpha\delta})} + \|v - v_h\|_{H^1(]0,2\pi[)}\Big) = 0.$$

**-H2** $\forall v \in H^1(\Omega), \exists (v_h)_{h>0} \in H^1(\Omega)$ such that

$$\lim_{h\to 0}\Big(\|v - v_h\|_{H^1(\Omega^+_{\alpha\delta})} + \|v - v_h\|_{H^1(\Omega^-_{\alpha\delta})} + \|v - v_h\|_{H^1(]0,2\pi[)}\Big) = 0.$$

**-H3** $F^\delta$ (defined by (67) is a bijection from $V^h_{\alpha\delta}$ to $V^h_0$.

We are interesting in the following approximate problem: find $u^\delta_h$ in $V^h_{\alpha\delta}$ such that

$$\forall v_h \in V^h_{\alpha\delta}, \qquad \breve{a}^\delta(u^\delta_h, v_h) = \int_{\Omega^+_{\alpha\delta}\cup\Omega^-_{\alpha\delta}} f v_h \, dx, \tag{86}$$

where $\breve{a}^\delta$ is defined by (75).

It is possible to prove a uniform discrete inf-sup condition:

**Proposition 4.1.** *There exist $\delta_0 > 0$, $h_0 > 0$ and a constant $C > 0$ such that*

$$\forall \delta < \delta_0, \forall h < h_0, \qquad \inf_{u_h \in V^h_{\alpha\delta}} \sup_{v_h \in V^h_{\alpha\delta}} \frac{\big|\breve{a}^\delta(u_h, v_h)\big|}{\|u_h\|_{\breve{V}_{\alpha\delta}}\|v_h\|_{\breve{V}_{\alpha\delta}}} \;\geq\; C. \tag{87}$$

The proof is similar to the proof of proposition 3.4 and proves the existence and uniqueness of $u^\delta_h$ for $\delta$ and $h$ small enough.
Using the discrete inf-sup condition (87), we obtain an uniform result of approximation:

**Proposition 4.2.** *Let $V^h_{\alpha\delta}$ be a finite dimensional subspace of $\breve{V}_{\alpha\delta}$ such that (H1),(H2) and (H3) hold. Then, there exist $\delta_0 > 0$ and $h_0 > 0$ and a constant $C > 1$ such that*

$$\forall \delta < \delta_0,\ \forall h < h_0 \qquad \|\breve{u}^\delta_1 - u^\delta_h\|_{\breve{V}^h_{\alpha\delta}} \leq C \inf_{v_h \in V^h_{\alpha\delta}} \|\breve{u}^\delta_1 - v_h\|_{\breve{V}^h_{\alpha\delta}}, \tag{88}$$

*where $\breve{u}^\delta_1$ is the solution of the continuous approximate problem (74) of and $u^\delta_h$ the solution of the discrete problem (86).*

*Proof.* By triangular inequality,

$$\|u - u^\delta_h\|_{V^h_{\alpha\delta}} \leq \|u - v_h\|_{V^h_{\alpha\delta}} + \|u^\delta_h - v_h\|_{V^h_{\alpha\delta}} \qquad \forall v_h \in V^h_{\alpha\delta}. \tag{89}$$

Moreover, since $\breve{a}^\delta(u - u^\delta_h, v_h) = 0 \ \forall v_h \in V^h_{\alpha\delta}$,

$$\breve{a}^\delta(v_h - u^\delta_h, w_h) = \breve{a}^\delta(v_h - u, w_h) \qquad \forall w_h \in V^h_{\alpha\delta}, \forall v_h \in V^h_{\alpha\delta}.$$

Choosing $w_h = v_h - u^\delta_h$ and combining the $\delta$-uniform continuity of $\breve{a}$ (for the norm $V_{\alpha\delta}$) with the discrete inf-sup condition (87) gives

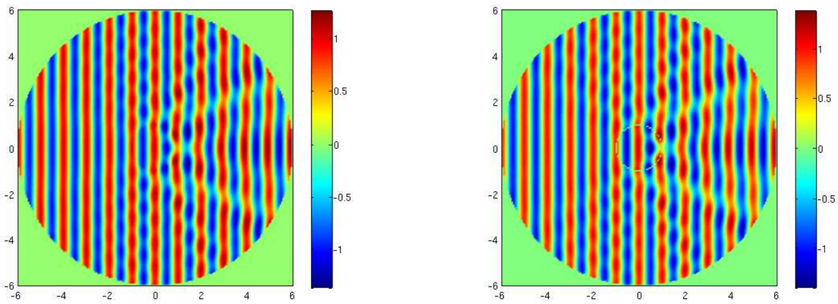
$$\|v_h - u^\delta_h\|_{\breve{V}_{\alpha\delta}} \leq C\|v_h - u\|_{\breve{V}_{\alpha\delta}} \qquad \forall v_h \in V^h_{\alpha\delta}. \tag{90}$$

Combining (89) with (90) gives the desired result. □

**Qualitative Results**

First the results are qualitatively good. In the figures 9 and 10, we can see that the approximate solution is close to the exact solution: the relative error is equal to $0.05$. The exact solution is computed with a strongly refined mesh.

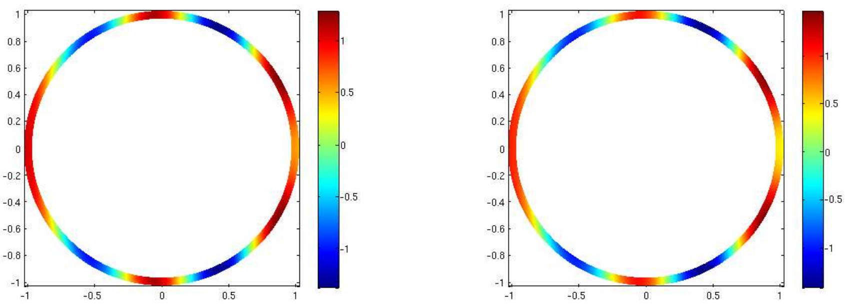
- Far field:




(a) ’Exact solution’ (b) Approximate solution

Figure 9: Comparison between the ’exact’solution and the far field approximate solution (N= 160)

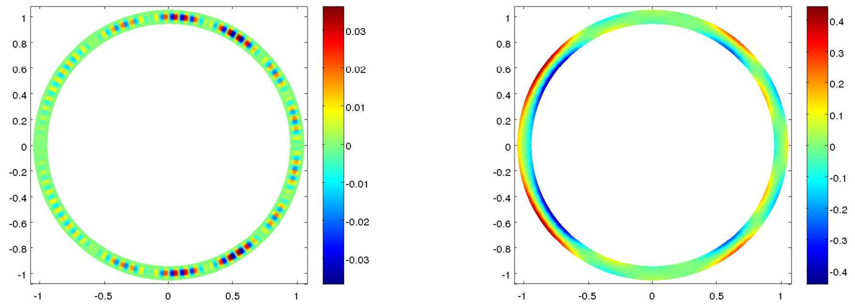
- Near field: As expected, the near field does not really oscillate since $U_0 = \langle u_0 \rangle$ does not depend on $S$.




(a) ’Exact solution’ (b) Approximate solution

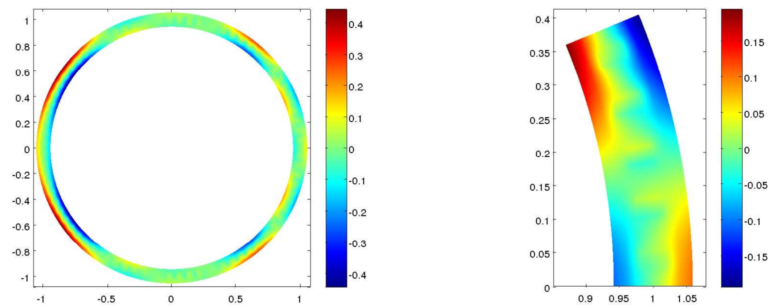
Figure 10: Comparison between the ’exact’solution and the near field approximate solution (N=160)

However, we can make visible the fast oscillations by plotting $\delta U_1 \approx \delta \left( \langle \frac{\partial u_1^\delta}{\partial \theta} \rangle V_1^1 + \langle \frac{\partial u_1^\delta}{\partial r} \rangle W_0^0 \right)$ (figure 12).



(a) $\delta \langle \frac{\partial u_1^\delta}{\partial \theta} \rangle V_1^1$ (b) $\delta \langle \frac{\partial u_1^\delta}{\partial r} \rangle W_0^0$

Figure 11: Functions $V_1^1$ and $W_0^0$



(a) $\delta U_1$ (b) $\delta U_1$

Figure 12: Reconstruction of $\delta U_1$

#### 4.2.1 Numerical Convergence Rates

It is clear that

$$\|u^\delta - u_h^\delta\|_{H^1(\Omega_\gamma)} \leq \|u^\delta - \breve{u}_1^\delta\|_{H^1(\Omega_\gamma)} + \|\breve{u}_1^\delta - u_h^\delta\|_{H^1(\Omega_\gamma)}.$$

The total error can be divided into two parts:

- The model error $\|u^\delta - \breve{u}_1^\delta\|_{H^1(\Omega_\gamma)}$ which is predominant when $h \ll \delta$. This error is expected to be proportional to $\delta^2$.
- The approximation error $\|\breve{u}_1^\delta - u_h^\delta\|_{H^1(\Omega_\gamma)}$ which is predominant when $h \gg \delta$.

Using a strongly refined mesh, it is possible to study the model error with respect to $\delta$. In the figure 13, we can see the $L^2$ and $H^1$ norms of the error with respect to $\delta$ in a logarithmic scale: we obtain a line of mean slope almost equal to 2 which is the expected convergence rate.
On the contrary, with a coarse mesh, we can study the approximation error. In the figure 14, we can see that the $L^2$ error with respect to the characteristic length $h$ of the mesh is almost independent of $\delta$. There is a priori no numerical locking.

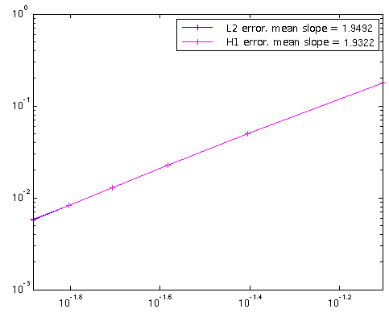



Figure 13: $L^2$ and $H^1$ error versus $\delta$ (logarithmic scale): N varies from 80 to 560

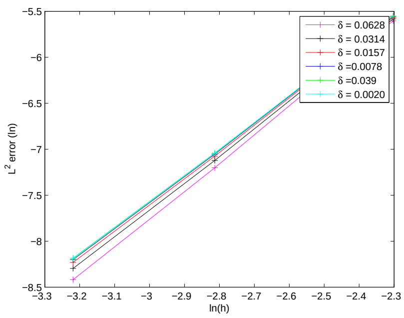



Figure 14: $L^2$ error according to $h$ (logarithmic scale)

# A Proofs of Technical Results

## A.1 Proof of Proposition 1.1

**Proposition 1.1.** *Problem (7) is well-posed. Moreover, there exists a constant C independent of* $\delta$ *such that,*

$$\|u^\delta\|_{H^1(\Omega)} \leq C \sup_{v \in H^1(\Omega), v \neq 0} \frac{|a^\delta(u^\delta, v)|}{\|v\|_{H^1(\Omega)}} \quad \forall u^\delta \in H^1(\Omega). \quad (8)$$

*Proof.* Direct proof of existence and uniqueness of the solution can be found in [24]. We only prove by contradiction the stability result (the proof is classical, similar kind of proof can be found in [7]). Assume that (8) is false. Then we could find a sequence $u^\delta$ such that

$$\|u^\delta\|_{H^1(\Omega)} = 1 \quad (a) \quad \text{and} \quad \lim_{\delta \to 0} \sup_{v \in H^1(\Omega), v \neq 0} \frac{|L^\delta(v)|}{\|v\|_{H^1(\Omega)}} = 0 \quad (b).$$

Since $u^\delta$ is bounded in $H_1(\Omega)$, there exist $u \in H_1(\Omega)$ and a sub-sequence (we also denote it $u^\delta$) such that, $u^\delta$ weakly tends to $u$ in $H^1(\Omega)$ when $\delta$ tends to $0$.
To be more precise, we also have

$$u^\delta \to u \quad \text{(strongly) in } L^2(\Omega) \qquad \text{and} \qquad u^\delta \rightharpoonup u \quad \text{(weakly) in } H^{\frac{1}{2}}(S_R).$$

Furthermore, for any $v \in H^1(\Omega)$,

$$\mu^\delta \nabla v \to \mu_\infty \nabla v \quad \text{(strongly) in } L^2(\Omega), \qquad \text{and} \qquad \rho^\delta v \to \rho_\infty v \quad \text{(strongly) in } L^2(\Omega).$$

It follows that

$$\lim_{\delta \to 0} a^\delta(u^\delta, v) = a_0(u, v), \quad (91)$$

where

$$a_0(u,v) = \int_\Omega \left(\mu_\infty \nabla u \cdot \nabla \bar{v} - \omega^2 \rho_\infty u \bar{v}\right) + \int_{s_R} \mu_\infty i\omega u \bar{v}.$$

But combining (91) with the assumption (b) gives

$$a_0(u,v) = 0 \quad \forall v \in H^1(\Omega).$$

The last assertion is nothing but the statement that $u$ is the variational solution of an homogeneous Helmholtz problem. Therefore $u = 0$ and $\lim\limits_{\delta\to 0} \|u^\delta\|_{L^2(\Omega)} = 0$.

Moreover, since $u^\delta$ is solution of the Helmholtz equation,

$$\mu_m \|\nabla u^\delta\|^2_{L^2(\Omega)} \leq |a^\delta(u^\delta, u^\delta)| + \omega^2 \rho_M \|u^\delta\|^2_{L^2(\Omega)}.$$

Letting $\delta$ tends to $0$ contradicts the assumption (a):

$$\lim_{\delta\to 0} \|u^\delta\|_{H^1(\Omega)} = 0.$$

□

## A.2 Proof of Proposition 2.1

**Proposition 2.1.** *Let $U_n$ be a function in $\mathcal{C}^\infty\left(]0,2\pi[\right) \times \mathcal{C}^\infty\left(\left\{(\mathcal{V},S) \in \mathbb{R}^2 \text{ such that } |\mathcal{V}| > \dfrac{1}{2}\right\}\right)$ which satisfies (19), which is 1-periodic in $S$ and which is non-exponentially increasing for large $\mathcal{V}$. The behaviour of $U_n$ for large $\mathcal{V}$ is given by*

$$U_n(\mathcal{V},S,\theta) = \sum_{k=0}^{n+1} C^+_{n,k}(\theta)\,\mathcal{V}^k \;+\; \sum_{l\in\mathbb{Z},l\neq 0} \Big(\sum_{k=0}^{n} B^+_{n,l,k}(\theta)\mathcal{V}^k\Big)\, e^{-2\pi|l|\mathcal{V}}\; e^{2i\pi l S} \qquad \text{for } \mathcal{V} > \frac{1}{2}, \tag{20}$$

$$U_n(\mathcal{V},S,\theta) = \sum_{k=0}^{n+1} C^-_{n,k}(\theta)\,\mathcal{V}^k \;+\; \sum_{l\in\mathbb{Z},l\neq 0} \Big(\sum_{k=0}^{n} B^-_{n,l,k}(\theta)\mathcal{V}^k\Big)\, e^{2\pi|l|\mathcal{V}}\; e^{2i\pi l S} \qquad \text{for } \mathcal{V} < -\frac{1}{2}, \tag{21}$$

*where $B^\pm_{n,l,k}(\theta)$ denote some constants that only depend on $\theta$.*

*Proof.* To shorten notation, it is useful to introduce the operators $\mathcal{A}^f_{i,l}$ (which are the Fourier transform of the operators $\mathcal{A}_i$ for $|\mathcal{V}| > \dfrac{1}{2}$): $\forall l \in \mathbb{Z}$,

$$\mathcal{A}^f_{0,l} = R_0^2 \mu_\infty \left( \frac{\partial^2}{\partial \mathcal{V}^2} - (2\pi l)^2 I_d \right),$$

$$\mathcal{A}^f_{1,l} = \mathcal{A}^{\theta,f}_{1,l} (\frac{\partial}{\partial \theta}) + \mathcal{A}^{0,f}_{1,l},$$

$$\mathcal{A}^{\theta,f}_{1,l} = 2i\pi l \; R_0 \; \mu_\infty I_d,$$

$$\mathcal{A}^{0,f}_{1,l} = \frac{1}{\mathcal{V}} \left( 2R_0 \mu_\infty \mathcal{V}^2 \frac{\partial^2}{\partial \mathcal{V}^2} + \mathcal{V} \frac{\partial}{\partial \mathcal{V}} \right),$$

$$\mathcal{A}^f_{2,l} = \mathcal{A}^{\theta\theta,f}_{2,l} (\frac{\partial^2}{\partial \theta^2}) + \mathcal{A}^{0,f}_{2,l},$$

$$\mathcal{A}^{\theta\theta,f}_{2,l} = \mu_\infty I_d,$$

$$\mathcal{A}^{0,f}_{2,l} = \mu_\infty \left( \mathcal{V}^2 \frac{\partial^2}{\partial \mathcal{V}^2} + R_0 \mathcal{V} \frac{\partial}{\partial \mathcal{V}} \right) + \omega^2 \rho_\infty R_0^2 I_d,$$

$$\mathcal{A}^f_{3,l} = \mathcal{V}(2R_0 \omega^2 \rho_\infty) I_d,$$

$$\mathcal{A}^f_{4,l} = \mathcal{V}^2 (\omega^2 \rho_\infty) I_d. \tag{92}$$

The proposition is proved by induction for $\mathcal{V} > \dfrac{1}{2}$ and similarly works for $\mathcal{V} < -\dfrac{1}{2}$.

1. Initialization of the induction for $n = 0$. Since $U_0$ is 1-periodic in $S$ and verifies an homogeneous Helmholtz equation if $\mathcal{V} > \dfrac{1}{2}$, $U_0$ can be written as a Fourier Series

$$U_0 := \sum_{l \in \mathbb{Z}} (U_0)^+_l(\mathcal{V},\theta) \; e^{2i\pi l Z}.$$

Then, we substitute this modal expansion in the near fields equation (19).

- For $l \neq 0$, we obtain

$$0 = \mathcal{A}^f_{0,l} (U_0)^+_l = R_0^2 \mu_\infty \left( \frac{\partial^2 (U_0)^+_l}{\partial \mathcal{V}^2} - l^2 (U_0)^+_l \right).$$

It follows that

$$(U_0)^+_l = B^+_{0,l}(\theta) \; e^{-2\pi |l| \mathcal{V}} \; + \; D^+_{0,l}(\theta) \; e^{2\pi |l| \mathcal{V}}.$$

Since $(U_0)^+_l$ is no-exponentially increasing, $D = 0$. Finally we have

$$(U_0)^+_l = B^+_{0,l}(\theta) \; e^{-2\pi |l| \mathcal{V}}.$$

- For $l = 0$, we have

$$0 = \mathcal{A}^f_{0,l} (U_0)^+_0 = \mu_\infty R_0^2 \frac{\partial^2 (U_0)^+_0}{\partial \mathcal{V}^2}.$$

Therefore,

$$(U_0)^+_0 = C^+_{0,0}(\theta) \; + \; \mathcal{V} \; C^+_{0,1}(\theta).$$

So, we have established the desired formula (20) for $n = 0$ and the modal expansion of $U_0$ is given by:

$$U_0(\mathcal{V}, S, \theta) = C^+_{0,0}(\theta) \; + \; \mathcal{V} \, C^+_{0,1}(\theta) \; + \sum_{l \in \mathbb{Z}^*} B^+_{0,l,0}(\theta) e^{-2\pi|l|\mathcal{V}} + o(\mathcal{V}^{-\infty})$$

2. Induction: assume the properties (20) and (21) hold for $k \leq n$; we shall prove it for $n+1$. First, as for $n = 0$, it possible to decompose $U_{n+1}$ as a Fourier series:

$$U_{n+1} := \sum_{l \, \in \mathbb{Z}} (U_{n+1})^+_l(\mathcal{V}, \theta) \; e^{2i\pi l Z}.$$

Then again, we substitute this modal expansion in the near fields equation (19).

- For $l \neq 0$, $U_{n+1,l}$ satisfies the following equation

$$\mathcal{A}^f_{0,l}(U_{n+1})^+_l = -\sum_{j=1}^{4} \mathcal{A}^f_{j,l}(U_{n+1-j})^+_l.$$

where the terms $B^f_{j,l}(U_{n+1-j})^+_l$ are polynomials in $\mathcal{V}$ and their degrees are smaller than $n$. Moreover $(U_{n+1})^+_l$ (like $(U_0)^+_l$) is not allowed to have an exponential increasing. It follows that

$$(U_{n+1})^+_l = \sum_{k=0}^{n+1} B_{n,l,k}(\theta)^+ \mathcal{V}^k e^{-2\pi|l|\mathcal{V}}.$$

Note that $B_{n,l,k}$ is determined for all $k > 0$ but is undetermined for $k = 0$.

- For $l = 0$, $(U_{n+1})^+_0$ satisfies the following equation

$$\mathcal{A}^f_{0,0}(U_{n+1})^+_0 = -\sum_{j=1}^{4} \mathcal{A}^f_{j,0}(U_{n+1-j})^+_0.$$

Since $(U_{n+1-j})^+_0$ is a polynomial of degree $n+2-j$, $\mathcal{A}^f_{j,0}(U_{n+1-j})^+_0$ is a polynomial of degree $n$ (indeed, we use the fact that if $P$ is a polynomial of degree $q$, thus $\mathcal{A}^f_{j,0}P$ is a polynomial of degree $q+j-2$ (see Remark B.2 and Proposition B.1 for more details)). Consequently,

$$(U_{n+1})^+_0 = \sum_{k=0}^{n+1} C_{n+1,k}(\theta)^+ \mathcal{V}^k,$$

and the proof is complete.

□

## A.3 Proof of Proposition 2.3

**Proposition 2.3.** *$\mathcal{W}(\mathbb{R}^2)$ is a Hilbert space. Moreover the semi-norm* $u \mapsto \left( \int_{B_0} |\nabla U|^2 \right)^{1/2}$ *is an equivalent norm on $\mathcal{W}(\mathbb{R}^2)$.*

*Proof.* Let $V$ be a function of $W^1(\mathbb{R}^2)$ and $\chi$ be a smooth truncated function which depends only on $\mathcal{V}$ such that

$$\chi(\mathcal{V}) := \begin{cases} 1 & \text{if } |\mathcal{V}| \leq 1, \\ 0 & \text{if } |\mathcal{V}| \geq 2. \end{cases}$$

We also consider $B_0^2 = \{(\mathcal{V},S) \in B_0, |\mathcal{V}| < 2\}$ and $< V >$ the mean value of $V$ in $B_0^2$

$$< V > := \frac{1}{4}\int_{-2}^{2}\int_{-1/2}^{1/2} V \; dS \; d\mathcal{V}.$$

We have

$$V(\mathcal{V},S)- < V > = \; (1-\chi)(\mathcal{V}) \; (V(\mathcal{V},S)- < V >) \; + \; \chi(\mathcal{V}) \; (V(\mathcal{V},S)- < V >).$$

The main idea of the proof is to separately dominate $\left\| \dfrac{\chi(V- < V >)}{(1+\mathcal{V}^2)^{1/2}} \right\|_{L^2(B_0)}$ and $\left\| \dfrac{(1-\chi)(V- < V >)}{(1+\mathcal{V}^2)^{1/2}} \right\|_{L^2(B_0)}$.
For the first upper bound, we can use the Poincare-Wirtinger inequality (see [25]) in $B_0^2$ to obtain

$$\left\| \frac{V- < V >}{(1+\mathcal{V}^2)^{1/2}} \right\|_{L^2(B_0^2)} \leq \| V- < V > \|_{L^2(B_0^2)} \leq C \left\| \nabla V \right\|_{L^2(B_0^2)}.$$

Consequently, we deduce that

$$\left\| \frac{\chi(V- < V >)}{(1+\mathcal{V}^2)^{1/2}} \right\|_{L^2(B_0)} \leq \left\| \frac{V- < V >}{(1+\mathcal{V}^2)^{1/2}} \right\|_{L^2(B_0^2)} \leq C \left\| \nabla V \right\|_{L^2(B_0)}.$$

For the second estimate, we introduce $\tilde{V}(\mathcal{V},S) \; := \; (1-\chi)(\mathcal{V}) \; (V(\mathcal{V},S)- < V >)$. We remark that $\tilde{V}(2,S) = 0$. Then we can use the Hardy Inequality

$$\begin{aligned}
\int_1^{+\infty} \frac{|\tilde{V}|^2}{\mathcal{V}^2} d\mathcal{V} &= \int_2^{+\infty} |\tilde{V}^2| (\frac{-1}{\mathcal{V}})' d\mathcal{V}, \\
&\leq \int_1^{+\infty} \frac{1}{\mathcal{V}} (\frac{\partial \tilde{V}}{\partial \mathcal{V}} \bar{\tilde{V}} + \tilde{V} \frac{\partial \bar{\tilde{V}}}{\partial \mathcal{V}}) d\mathcal{V}, \\
&\leq 2 \int_1^{+\infty} \frac{1}{\mathcal{V}} Re(\frac{\partial \tilde{V}}{\partial \mathcal{V}} \bar{\tilde{V}}) d\mathcal{V}, \\
&\leq 2 \left( \int_1^{+\infty} |\frac{\tilde{V}^2}{\mathcal{V}^2}| d\mathcal{V} \right)^{1/2} \left( \int_1^{+\infty} |\nabla \tilde{V}|^2 d\mathcal{V} \right)^{1/2}.
\end{aligned}$$

Moreover,

$$\int_1^{+\infty} \frac{|\tilde{V}|^2}{1+\mathcal{V}^2} d\mathcal{V} \leq \int_1^{+\infty} \frac{|\tilde{V}|^2}{\mathcal{V}^2} d\mathcal{V}.$$

It follows that

$$\int_{-1/2}^{1/2}\int_1^{+\infty} \frac{|\tilde{V}|^2}{1+\mathcal{V}^2} d\mathcal{V} dS \leq 4 \int_{-1/2}^{1/2}\int_1^{+\infty} |\nabla \tilde{V}|^2 d\mathcal{V} dS.$$

Therefore,

$$\begin{aligned}
\left\| \frac{(1-\chi)(V- < V >)}{(1+\mathcal{V}^2)^{1/2}} \right\|_{L^2(B_0)} &\leq C \left( \left\| \nabla V \right\|_{L^2(B_0)} + \left\| \chi' \right\|_{L^\infty(\mathbb{R})} \left\| (V- < V >) \right\|_{L^2(B_0^2)} \right), \\
&\leq C \left\| \nabla V \right\|_{L^2(B_0)}
\end{aligned}$$

To end the prove, it suffices to observe that

$$\inf_{c \in \mathbb{C}} \left\| \frac{V-c}{(1+\mathcal{V}^2)^{1/2}} \right\|_{L^2(B_0)} \leq \left\| \frac{V- < V >}{(1+\mathcal{V}^2)^{1/2}} \right\|_{L^2(B_0)} \leq C \left\| \nabla V \right\|_{L^2(B_0)}.$$

□

## A.4 Proof of Proposition 2.10

**Proposition 2.10.** *There is a constant $C$ independent of $\delta$ such that*

$$|a^\delta(\varepsilon_n^\delta, v)| \leq C(\eta^{n-\frac{1}{2}} + \delta^{n-1})\|v\|_{H^1(\Omega)} \tag{58}$$

*Proof.* Our proof starts with the observation that

$$a^\delta(\varepsilon_n^\delta, v) = a^\delta(u^\delta, v) - a^\delta((1-\chi_\eta)u_e^n, v) - a^\delta(\chi_\eta U_i^n, v).$$

Moreover, an immediate verification shows that

$$\int_\Omega \mu^\delta(\nabla(1-\chi_\eta)u_e^n)\cdot\nabla\bar{v} = \int_\Omega \mu^\delta\nabla u_e^n\cdot\nabla((1-\chi_\eta)\bar{v}) - \int_\Omega \mu^\delta\nabla u_e^n\cdot\nabla(1-\chi_\eta)\bar{v} + \int_\Omega \mu^\delta u_e^n\nabla(1-\chi_\eta)\cdot\nabla\bar{v}.$$

Therefore

$$a^\delta(\varepsilon_n^\delta, v) = \underbrace{\int_\Omega \mu^\delta(u_e^n - U_i^n)\nabla\chi_\eta\cdot\nabla\bar{v} - \mu^\delta(\nabla(u_e^n - U_i^n)\cdot\nabla\chi_\eta)\bar{v}}_{\text{matching error}} + \underbrace{a(U_i^n, \chi_\eta v)}_{\text{equation error}}.$$

The consistency error $|a^\delta(\varepsilon_n^\delta, v)|$ is divided into two distinct parts, that we have to bound separately :

- a first term due to the matching error

$$L_M^\eta(v) := \int_\Omega \mu^\delta(u_e^n - U_i^n)\nabla\chi_\eta\cdot\nabla\bar{v} - \mu^\delta(\nabla(u_e^n - U_i^n)\cdot\nabla\chi_\eta)\bar{v},$$

- a second term due to the error on Helmholtz equation for the near fields

$$L_{NF}^\eta(v) := a(U_i^n, \chi_\eta v).$$

1. **Estimate of the matching error**:
   Le us denote by $C_\eta^\delta$ the support of $\nabla\chi_\eta$

$$C_\eta^\delta := \left\{(r,\theta) \ \in \ \mathbb{R}^+\times[0,2\pi] \quad \text{such that} \quad \eta \leq |r-R_0| \leq 2\eta\right\}.$$

   Note that $C_\eta^\delta$ is included in the overlapping zones. The matching error can be written as

$$L_M^\eta(v) := \int_{C_\eta^\delta} \mu^\delta(u_e^n - U_i^n)\nabla\chi_\eta\cdot\nabla\bar{v} - \mu^\delta(\nabla(u_e^n - U_i^n)\cdot\nabla\chi_\eta)\bar{v}.$$

   By regularity of far and near fields, we bound $u_e^n$ and $U_i^n$ by their $L^\infty$ norm,

$$|L_M^n(v)| \leq \Big(\Big(\|u_e^n - U_i^n\|_{L^\infty(C_\eta^\delta)} + \|\nabla(u_e^n - U_i^n)\|_{L^\infty(C_\eta^\delta)}\Big)\|\nabla\chi_\eta\|_{L^\infty(C_\eta^\delta)}\|\mu\|_{L^\infty(\Omega)} (\|v\|_{L^1(C_\eta^\delta)} + \|\nabla v\|_{L^1(C_\eta^\delta)})\Big). \tag{93}$$

   Moreover, taking into account that $C_\eta^\delta$ is bounded (the measure of $(C_\eta^\delta)$ is proportional to $\eta$), we have

$$(\|v\|_{L^1(C_\eta^\delta)} + \|\nabla v\|_{L^1(C_\eta^\delta)}) \leq C\eta^{1/2}\|v\|_{H^1(C_\eta^\delta)} \tag{94}$$

and

$$\|\nabla\chi_\eta\|_{L^\infty(C_\eta^\delta)} \leq \frac{1}{\eta}\|\nabla\chi\|_{L^\infty(\mathbb{R})}. \tag{95}$$

Combining (93),(94) and (95) yields

$$|L_M^n(v)| \leq C\eta^{-1/2}\left(\|u_e^n - U_i^n\|_{L^\infty(C_\eta^\delta)} + \|\nabla(u_e^n - U_i^n)\|_{L^\infty(C_\eta^\delta)}\right)\|v\|_{H^1(\Omega)}. \tag{96}$$

What is left is to evaluate $\|u_e^n - U_i^n\|_{L^\infty(C_\eta^\delta)}$ and $\|\nabla(u_e^n - U_i^n)\|_{L^\infty(C_\eta^\delta)}$ using of course the matching conditions. For convenience we use the form (24) of these conditions.

By the integral form of Taylor's formula, we obtain

$$u_e^{n,\pm}(r,\theta) = \sum_{k=0}^{n}\delta^k\left(\sum_{i=0}^{n-k}\frac{(r-R_0)^i}{i!}\frac{\partial^i u_k^\pm(R_0,\theta)}{\partial r} + \int_{R_0}^{r}\frac{\partial^{n-k+1}u_k^\pm(t,\theta)}{\partial r^{n-k+1}}\frac{(r-t)^{n-k}}{(n-k)!}dt\right).$$

Moreover, since $\lim\limits_{\delta\to 0}\dfrac{\eta(\delta)}{\delta} = +\infty$, we can use modal expansions (20) and (21) for the near fields,

$$U_i^{n,\pm}(r,\theta) = \sum_{k=0}^{n}\delta^k\left(\sum_{i=0}^{n-k}\frac{(r-R_0)^i}{i!}\frac{\partial^i u_k^\pm(R_0,\theta)}{\partial r}\right) + R_n^\pm(r,\theta),$$

where $R_n^\pm(r,\theta)$ is an evanescent term, i.e

$$\forall N\in\mathbb{N},\ \exists\, C_N>0, \quad \|R_n\|_{L^\infty(C_\eta^\delta)} \leq C_N\delta^N.$$

Consequently,

$$u_e^n(r,\theta) - U_i^n(r,\theta) = \sum_{k=0}^{n}\delta^k\left(\int_{R_0}^{r}\frac{\partial^{n-k+1}u_k(t,\theta)}{\partial r^{n-k+1}}\frac{(r-t)^{n-k}}{(n-k)!}\right) + R_n(r,\theta).$$

Using again the regularity of $u_n$, it is possible to directly dominate $\dfrac{\partial^{n-k+1}u_k^\pm(t,\theta)}{\partial r^{n-k+1}}$ by a constant. We can now conclude

$$\|u_e^n(r,\theta) - U_i^n(r,\theta)\|_{L^\infty(C_\eta^{\delta+})} \leq C_n\sum_{k=0}^{n}C\delta^k\eta^{n-k+1} \leq C_n\eta^{n+1}. \tag{97}$$

In the same manner, we obtain an estimate on the gradient

$$\|\nabla(u_e^n - U_i^n)\|_{L^\infty(C_\eta^{\delta+})} \leq C_n\eta^n. \tag{98}$$

Combining (96) (97) and (98) we have the matching error estimate:

$$|L_M^n(v)| \leq C\eta^{n-1/2}\|\nabla v\|_{H^1(\Omega)}. \tag{99}$$

2. **Estimate of the Helmholtz equation error**

Set $\mathcal{C}^\eta := \left\{(x,y)\in R^2,\ \ R_0 - 2\eta(\delta) \leq \sqrt{x^2+y^2} \leq R_0 + 2\eta(\delta)\right\}$. The Helmholtz equation error $L_{NF}^\eta(v)$ is given by (note that the notation $\int_{C^\eta}$ is improper and should be replaced by duality),

$$\begin{aligned} L_{NF}^\eta(v) &:= a(U_i^n, \chi_\eta v), \\ &= \int_\Omega \mu^\delta\nabla U_i^n\cdot\nabla\bar{\chi_\eta v}\,dx - \int_\Omega \rho^\delta\omega^2 U_i^n\bar{v}\chi_\eta\,dx, \\ &= -\int_{C^\eta}(\nabla\cdot(\mu^\delta\nabla U_i^n) + \rho^\delta\omega^2 U_i^n)\bar{v}\chi_\eta\,dx. \end{aligned}$$

In addition, near field equations (19) give (with $\mathcal{V} = \frac{r-R_0}{\delta}$ and $S = \frac{\theta R_0}{\delta}$).

$$r^2 \left(\nabla \cdot (\mu^\delta \nabla U_i^n) + \rho^\delta \omega^2 U_i^n\right)(\mathcal{V}, S, \theta) = \sum_{j=-2}^{2} \sum_{i=0}^{n} \delta^{j+i} \mathcal{A}_{j+2}\left(U_i\right)(\mathcal{V}, S, \theta)$$

$$= \sum_{i=0}^{n-2} \delta^i \underbrace{\left( \sum_{j=-2}^{\min(i,2)} \mathcal{A}_{j+2}(U_{i-j})(\mathcal{V}, S, \theta) \right)}_{=0} + \sum_{i=n-1}^{n+2} \delta^i \left( \sum_{j=i-n}^{\min(i,2)} \mathcal{A}_{j+2}(U_{i-j})(\mathcal{V}, S, \theta) \right).$$

Therefore,

$$\begin{aligned} |L_{NF}^\eta(v)| &= \left| \sum_{i=n-1}^{n+2} \delta^i \int_{C^\eta} \mathcal{A}_{j+2}(U_{i-j}) \bar{v} \chi_\eta \, dx \right|, \\ &\leq \left| \sum_{i=n-1}^{n+2} \delta^i \int_{C^\eta} \mathcal{A}_{j+2}(U_{i-j}) \bar{v} \, dx \right|. \end{aligned}$$

We shall dominate $|\sum_{i=n-1}^{n+2} \delta^i \int_{C^\eta} \mathcal{A}_{j+2}(U_{i-j})\bar{v}|$ by separating it into two distinct parts:

$$\begin{aligned} &|\sum_{i=n-1}^{n+2} \delta^i \int_{C^\eta} \mathcal{A}_{j+2}(U_{i-j})\bar{v}| dx \leq \\ &\underbrace{|\sum_{i=n-1}^{n+2} \delta^i \int_0^{2\pi} \int_{R_0 \pm \delta}^{\pm 2\eta} \mathcal{A}_{j+2}(U_{i-j}) \bar{v} \; r \; dr \; d\theta|}_{A} + \underbrace{|\sum_{i=n-1}^{n+2} \delta^i \int_0^{2\pi} \int_{R_0-\delta}^{R_0+\delta} \mathcal{A}_{j+2}(U_{i-j}) \bar{v} \; r \; dr \; d\theta|}_{B}. \end{aligned} \tag{100}$$

**• Estimate of A**

When $r \in ]R_0 + \delta, 2\eta[$, $U_{i-j}$ and its derivatives of any order are bounded and $U_{i-j}$ has a polynomial behaviour of degree $i - j$ (Proposition 2.1). Moreover $\mathcal{A}_{j+2}$ is an homogeneous operator of order $j$ (see (92), Remark B.2 and Proposition B.1), which means in particular that if $U = \mathcal{V}^k + o(\mathcal{V}^{-\infty})$, then $\mathcal{A}_{j+2}(U) = C\mathcal{V}^{j+k} + o(\mathcal{V}^{-\infty})$. Therefore, $\mathcal{A}_{j+2}(U_{i-j})$ are in $L^\infty(]R_0 + \delta, R_0 + 2\eta[\times]0, 2\pi[)$ and

$$\|\mathcal{A}_{j+2}(U_{i-j})\|_{L^\infty(]R_0+\delta, R_0+2\eta[\times]0,2\pi[)} \leq C \left(\frac{\eta}{\delta}\right)^{i-j} \left(\frac{\eta}{\delta}\right)^j \leq C \left(\frac{\eta}{\delta}\right)^i.$$

Finally,

$$\begin{aligned} A = \left| \sum_{i=n-1}^{n+2} \delta^i \int_0^{2\pi} \int_{R_0-\delta}^{R_0+\delta} \mathcal{A}_{j+2}(U_{i-j}) \bar{v} \; r \; dr \; d\theta \right| &\leq \sum_{i=n-1}^{n+2} \delta^i \left(\frac{\eta}{\delta}\right)^i \|v\|_{L^1(]R_0+\delta, R_0+2\eta[\times]0,2\pi[)}, \\ &\leq C \sum_{i=n-1}^{n+2} \eta^{i+\frac{1}{2}} \|v\|_{L^2(\Omega)}, \\ &\leq C\eta^{n-\frac{1}{2}} \|v\|_{L^2(\Omega)}. \end{aligned} \tag{101}$$

• **Estimate of B**:

To shorten notation it is useful to consider truncated periodicity cell

$$\Omega_1 = \left\{ (\mathcal{V}, S) \in \mathbb{R}^2, \text{such that} - 1 \le \mathcal{V} \le 1 \text{ and } -\frac{1}{2} \le S \le \frac{1}{2} \right\}.$$

We remind that

$$U_n(\mathcal{V}, S, \theta) = \sum_{j=0}^{n} \sum_{k=0}^{j} \underbrace{\langle \frac{\partial^k u_{n-j}(R_0,\theta)}{\partial \theta^k} \rangle}_{\mathcal{C}^\infty_{per}(]0,2\pi[)} \underbrace{V_j^k(\mathcal{V}, S)}_{H^1(\Omega^1)} + \sum_{j=0}^{n-1} \sum_{k=0}^{j} \underbrace{\langle \frac{\partial^{k+1} u_{n-1-j}(R_0,\theta)}{\partial \theta^k \partial r} \rangle}_{\mathcal{C}^\infty_{per}(]0,2\pi[)} \underbrace{W_j^k(\mathcal{V}, S)}_{H^1(\Omega_1)}.$$

Therefore, to estimate $B$, it is sufficient to estimate the following generic quantities for $j \in \{1,2,3,4\}$:

$$\left| \int_0^{2\pi} \int_{R_0-\delta}^{R_0+\delta} \mathcal{A}_j \left( U(\mathcal{V},S,\theta) \right)_{|\mathcal{V}=\frac{r-R_0}{\delta}, S=\frac{R_0\theta}{\delta}} \; \bar{v} \; r \; dr \; d\theta \right|,$$

where,

$$U = g(\theta) V(S, \mathcal{V}), \quad g(\theta) \in \mathcal{C}^\infty_{per}(]0,2\pi[) \quad \text{and} \quad V \text{ is } 1-\text{periodic in } S.$$

We now explain in details the previous estimate for $j=1$.

$$\left| \int_0^{2\pi} \int_{R_0-\delta}^{R_0+\delta} \mathcal{A}_1 \left( U(\mathcal{V},S,\theta) \right)_{|\frac{r-R_0}{\delta}, \frac{R_0\theta}{\delta}} \; \bar{v} \; r \; dr \; d\theta \right| \le \underbrace{\left| \int_0^{2\pi} \int_{R_0-\delta}^{R_0+\delta} g'(\theta) \mathcal{A}_1^\theta \left( V(\mathcal{V},S) \right)_{|\frac{r-R_0}{\delta}, \frac{R_0\theta}{\delta}} \; \bar{v} \; r \; dr \; d\theta \right|}_{B_1^\theta}$$

$$+ \underbrace{\left| \int_0^{2\pi} \int_{R_0-\delta}^{R_0+\delta} g(\theta) \mathcal{A}_1^0 \left( V(\mathcal{V},S) \right)_{|\frac{r-R_0}{\delta}, \frac{R_0\theta}{\delta}} \; \bar{v} \; r \; dr \; d\theta \right|}_{B_1^0}.$$

- **Estimate of $B_1^\theta$**

  Since $|g'(\theta)| \le C \quad (g \in \mathcal{C}^\infty_{per}(]0,2\pi[))$,

$$B_1^\theta \le C \left| \int_0^{2\pi} \int_{R_0-\delta}^{R_0+\delta} \left( R_0 \frac{\partial}{\partial S}(\mu V) + \mu R_0 \frac{\partial V}{\partial S} \right)_{|S=\frac{R_0\theta}{\delta}, \mathcal{V}=\frac{r-R_0}{\delta}} \; \bar{v}(r,\theta) \; r \; dr \; d\theta \right|.$$

  We cut the angular integral into $N = \dfrac{2\pi R_0}{\delta}$ even parts:

$$B_1^\theta \le C \left| \sum_{k=0}^{\frac{2\pi R_0}{\delta}} \int_{\frac{k\delta}{R_0}}^{\frac{(k+1)\delta}{R_0}} \int_{R_0-\delta}^{R_0+\delta} \left( R_0 \frac{\partial}{\partial S}(\mu V) + \mu R_0 \frac{\partial V}{\partial S} \right)_{|S=\frac{R_0\theta}{\delta}, \mathcal{V}=\frac{r-R_0}{\delta}} \; \bar{v}(r,\theta) \; r \; dr \; d\theta \right|.$$

Applying the change of scale $S = \dfrac{R_0\theta}{\delta}$, $\mathcal{V} = \dfrac{r - R_0}{\delta}$ and dominating $r = R_0 + \delta\mathcal{V}$ by a constant give

$$
\begin{aligned}
B_1^\theta &\leq C\left|\sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \int_k^{k+1}\int_{-1}^{1}\left(R_0\frac{\partial}{\partial S}(\mu V) + \mu R_0\frac{\partial V}{\partial S}\right)(\mathcal{V},S,\tfrac{S\delta}{R_0})\ \bar{v}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})\ d\mathcal{V}\ dS\right|,\\
&\leq C\left(\underbrace{\left|\sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \int_k^{k+1}\int_{-1}^{1}\left(R_0\frac{\partial}{\partial S}(\mu V)\right)(\mathcal{V},S,\tfrac{S\delta}{R_0})\ \bar{v}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})\ d\mathcal{V}\ dS\right|}_{a}\right.\\
&\qquad\left.+\underbrace{\left|\sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \int_k^{k+1}\int_{-1}^{1}\left(\mu R_0\frac{\partial V}{\partial S}\right)(\mathcal{V},S,\tfrac{S\delta}{R_0})\ \bar{v}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})\ d\mathcal{V}\ dS\right|}_{b}\right).
\end{aligned}
$$

Again, we separately estimate $a$ and $b$.
By integration by part and using the fact that $\dfrac{\partial}{\partial S}\left(v(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})\right) = \dfrac{\delta}{R_0}\dfrac{\partial v}{\partial\theta}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})$, we have

$$
\begin{aligned}
a &\leq \underbrace{\left|\sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \int_{-1}^{1}\left(\mu V(\mathcal{V},k+1)\ \bar{v}(R_0+\delta\mathcal{V},\tfrac{(k+1)\delta}{R_0}) - \mu V(\mathcal{V},k)\ \bar{v}(R_0+\delta\mathcal{V},\tfrac{(k)\delta}{R_0})\right)\right|}_{=0}\\
&\qquad+\left|\sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \int_k^{k+1}\int_{-1}^{1} R_0\mu V\frac{\partial}{\partial S}\left(\bar{v}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})\right)\ d\mathcal{V}\ dS\right|.
\end{aligned}
$$

But, by Cauchy-Schwarz inequality,

$$
\begin{aligned}
\left|\int_k^{k+1}\int_{-1}^{1} R_0\mu V\frac{\partial}{\partial S}\left(\bar{v}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})\right)\ d\mathcal{V}\ dS\right| &\leq C\|V\|_{H^1(\Omega_1)}\left(\delta^2\int_k^{k+1}\int_{-1}^{1}\left|\frac{\partial v}{\partial\theta}\right|^2 d\mathcal{V}\ dS\right)^1,\\
&\leq C\|V\|_{H^1(\Omega_1)}\left(\int_{\frac{k\delta}{R_0}}^{\frac{(k+1)\delta}{R_0}}\int_{R_0-\delta}^{R_0+\delta}\left|\frac{\partial v}{\partial\theta}\right|^2\ r\ drd\theta\right)^1,\\
&\leq C\|V\|_{H^1(\Omega_1)}\|v\|_{H^1(\Omega)}.
\end{aligned}
$$

Therefore, we obtain an upper bound for $a$,

$$
\begin{aligned}
a &\leq C\sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2\|V\|_{H^1(\Omega_1)}\|v\|_{H^1(\Omega)},\\
&\leq C\ \delta\ \|v\|_{H^1(\Omega)}.
\end{aligned}
$$

In the same way,

$$\begin{aligned} b &\leq C \sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \|V\|_{H^1(\Omega_1)} \left( \int_k^{k+1} \int_{-\frac{1}{2}}^{\frac{1}{2}} |v|^2 \, d\mathcal{V} dS \right)^{\frac{1}{2}}, \\ &\leq C \sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta \|V\|_{H^1(\Omega_1)} \|v\|_{H^1(\Omega)}. \\ &\leq C \, \|v\|_{H^1(\Omega)} \end{aligned}$$

Finally,

$$B_1^\theta \leq C \, \|v\|_{H^1(\Omega)}.$$

- **Estimate of** $B_1^0$

As previously, we first divide the angular integral into $N = \dfrac{2\pi R_0}{\delta}$ parts, then we apply the change of scale $S = \dfrac{R_0 \theta}{\delta}$, $\mathcal{V} = \dfrac{r - R_0}{\delta}$:

$$\begin{aligned} B_1^0 &\leq C \left| \sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \int_k^{k+1} \int_{-1}^{1} \left( 2R_0 \mathcal{V} \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial V}{\partial \mathcal{V}} \right) + R_0 \mathcal{V} \mu \frac{\partial V}{\partial \mathcal{V}} \right) \, \bar{v}_{(R_0 + \delta \mathcal{V}, \frac{S\delta}{R_0})} \, d\mathcal{V} \, dS \right|, \\ &\leq C \left( \underbrace{\sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \left| \int_k^{k+1} \int_{-1}^{1} \left( 2R_0 \mathcal{V} \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial V}{\partial \mathcal{V}} \right) \right) \, \bar{v}_{(R_0 + \delta \mathcal{V}, \frac{S\delta}{R_0})} \, d\mathcal{V} \, dS \right|}_{a} \right. \\ &\qquad \left. + \underbrace{\sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \left| \int_k^{k+1} \int_{-1}^{1} R_0 \mathcal{V} \mu \frac{\partial V}{\partial \mathcal{V}} \, \bar{v}_{(R_0 + \delta \mathcal{V}, \frac{S\delta}{R_0})} \, d\mathcal{V} \, dS \right|}_{b} \right). \end{aligned}$$

To estimate $a$, it is useful to first dominate the following integral:

$$\begin{aligned} \left| \int_k^{k+1} \int_{-1}^{1} \left( 2R_0 \mathcal{V} \frac{\partial}{\partial \mathcal{V}} \left( \mu \frac{\partial V}{\partial \mathcal{V}} \right) \right) \bar{v} \right| &\leq \underbrace{\int_k^{k+1} \int_{-1}^{1} \left| 2R_0 \mu \frac{\partial V}{\partial \mathcal{V}} \, \bar{v}_{(R_0 + \delta \mathcal{V}, \frac{S\delta}{R_0})} \right| \, d\mathcal{V} \, dS}_{a_1} \\ &+ \underbrace{\int_k^{k+1} \int_{-1}^{1} \left| 2\mathcal{V} R_0 \mu \frac{\partial V}{\partial \mathcal{V}} \, \frac{\partial}{\partial \mathcal{V}} \left( \bar{v}_{(R_0 + \delta \mathcal{V}, \frac{S\delta}{R_0})} \right) \right| \, d\mathcal{V} \, dS}_{a2} \\ &+ \underbrace{\int_k^{k+1} \left| \mu \frac{\partial V}{\partial \mathcal{V}}(S, 1) \, \bar{v}_{(R_0 + \delta, \frac{S\delta}{R_0})} \right| dS}_{a_3^+} \\ &+ \underbrace{\int_k^{k+1} \left| \mu \frac{\partial V}{\partial \mathcal{V}}(S, -1) \, \bar{v}_{(R_0 - \delta, \frac{S\delta}{R_0})} \right| dS}_{a_3^-}. \end{aligned}$$

The terms $a_1$ and $a_2$ are classical and can be treated as previously:

$$
\begin{aligned}
a_1 &\leq C\|V\|_{H^1(\Omega_1)} \left( \int_k^{k+1} \int_{-1}^{1} |v|^2 dS d\mathcal{V} \right)^1, \\
&\leq \frac{C}{\delta} \|V\|_{H^1(\Omega_1)} \|v\|_{H^1(\Omega)}, \\
&\leq \frac{C}{\delta} \|v\|_{H^1(\Omega)}.
\end{aligned}
$$

$$
\begin{aligned}
a_2 &\leq C\|V\|_{H^1(\Omega_1)} \left( \delta^2 \int_k^{k+1} \int_{-1}^{1} \left| \frac{\partial v}{\partial r} \right|^2 dS d\mathcal{V} \right)^{\frac{1}{2}} \\
&\leq C\|v\|_{H^1(\Omega)}.
\end{aligned}
$$

To estimate $a_3^{\pm}$ we first remark that $\mu \dfrac{\partial V}{\partial \mathcal{V}}(S, \pm 1)$ are in $L^\infty(]k, k+1[)$. Then we use the trace theorem,

$$
\begin{aligned}
a_3^{\pm} &\leq C \left( \int_k^{k+1} |v(R_0 \pm \delta, \frac{S\delta}{R_0})\ dS|^2 \right)^{\frac{1}{2}}, \\
&\leq \frac{C}{\sqrt{\delta}} \|v\|_{H^1(\Omega)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
a &\leq \sum_{k=0}^{\frac{2\pi R_0}{\delta}} (a_1 + a_2 + a_3^+ + a_3^-), \\
&\leq C\|v\|_{H^1(\Omega)}.
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
b &\leq C \sum_{k=0}^{\frac{2\pi R_0}{\delta}} \delta^2 \left( \frac{1}{\delta} \|v\|_{H^1(\Omega)} \|V\|_{H^1(\Omega_1)} \right) \\
&\leq C\|v\|_{H^1(\Omega)}
\end{aligned}
$$

Therefore,

$$
B_1^0 \leq C\|v\|_{H^1(\Omega)},
$$

and

$$
\int_0^{2\pi} \int_{R_0-\delta}^{R_0+\delta} \mathcal{A}_1(U) \bar{v}\ r\ dr\ d\theta \leq\ C\ \|v\|_{H^1(\Omega)}.
$$

• We can now proceed analogously to prove:

$$\int_0^{2\pi}\int_{R_0-\delta}^{R_0+\delta} \mathcal{A}_2(U)\bar{v}\; r\; dr\; d\theta \leq\; C\;\|v\|_{H^1(\Omega)},$$

$$\int_0^{2\pi}\int_{R_0-\delta}^{R_0+\delta} \mathcal{A}_3(U)\bar{v}\; r\; dr\; d\theta \leq\; C\;\|v\|_{H^1(\Omega)},$$

$$\int_0^{2\pi}\int_{R_0-\delta}^{R_0+\delta} \mathcal{A}_4(U)\bar{v}\; r\; dr\; d\theta \leq\; C\;\|v\|_{H^1(\Omega)}.$$

Finally,

$$\begin{aligned} B &= \left|\sum_{i=n-1}^{n+2}\int_0^{2\pi}\int_{R_0-\delta}^{R_0+\delta} \mathcal{A}_{j+2}(U_{i-j})\bar{v}\; r\; dr\; d\theta\right|, \\ &\leq\; C\;\delta^{n-1}\;\|v\|_{H^1(\Omega)}. \end{aligned} \tag{102}$$

Combining (101) and (102) with inequality (100) we obtain an estimate of the error on the Helmholtz equation:

$$\begin{aligned} |a(U_i^n,\chi_\eta v)| &\leq\; C\;\left(\eta^{n-\frac{1}{2}}+\delta^{n-1}\right)\;\|v\|_{H^1(\Omega)}, \\ &\leq\; C\;\eta^{n-1}\;\|v\|_{H^1(\Omega)}. \end{aligned} \tag{103}$$

Adding (103) and (99) gives the desired conclusion (58). □

## A.5 Proof of Estimates (62) of Proposition 2.11

### A.5.1 $\left(\int_{-\gamma}^{\gamma}\int_{\mathbf{j}}^{\mathbf{j+1}}\left|\mathbf{u}^\delta(\mathbf{R_0}+\delta\mathcal{V},\frac{\mathbf{S}\delta}{\mathbf{R_0}})-\sum_{\mathbf{k=0}}^{\mathbf{n}}\delta^{\mathbf{k}}\mathbf{U_k}(\mathcal{V},\mathbf{S},\frac{\mathbf{S}\delta}{\mathbf{R_0}})\right|^{\mathbf{2}}\mathbf{dS}\;\mathbf{d}\mathcal{V}\right)^{\frac{1}{2}}\leq\mathbf{C}\delta^{\mathbf{n+1}}$

Let us define $A$ by

$$A=\int_{-\gamma}^{\gamma}\int_j^{j+1}\left|u^\delta(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})-\sum_{k=0}^{n}\delta^kU_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right|^2 dS\;d\mathcal{V}.$$

It is clear that, $\forall P\in\mathbb{N}$

$$A\leq C\left(\int_{-\gamma}^{\gamma}\int_j^{j+1}\left|u^\delta(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})-\sum_{k=0}^{n+P}\delta^kU_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right|^2 dS\;d\mathcal{V}+\sum_{k=n+1}^{n+P}\delta^{2k}\int_{-\gamma}^{\gamma}\int_j^{j+1}\left|U_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right|^2 dS\;d\mathcal{V}\right).$$

But,

$$\int_{-\gamma}^{\gamma}\int_j^{j+1}\left|U_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right|^2 dS\;d\mathcal{V}\leq C.$$

Consequently,

$$\sum_{k=n+1}^{n+P}\delta^k\int_{-\gamma}^{\gamma}\int_j^{j+1}\left|U_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right|^2 dS\;d\mathcal{V}\leq C\delta^{2(n+1)}. \tag{104}$$

Moreover,

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|u^{\delta}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})-\sum_{k=0}^{n+P}\delta^k U_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right|^2 dS\; d\mathcal{V} \leq$$

$$\frac{C}{\delta^2}\int_{R_0-\gamma\delta}^{R_0+\gamma\delta}\int_{\frac{k\delta}{R_0}}^{\frac{(k+1)\delta}{R_0}}\left|u^{\delta}(r,\theta)-\sum_{k=0}^{n+P}\delta^k U_k(\tfrac{r-R_0}{\delta},\tfrac{R_0\theta}{\delta},\theta)\right|^2 r\; dr\; d\theta.$$

For $\delta$ small enough,in $]R_0-\gamma\delta,R_0+\gamma\delta[\times]0,2\pi[$), $u^{\delta}(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})-\sum_{k=0}^{n+P}\delta^k U_k(\mathcal{V},S,\frac{S\delta}{R_0})=\varepsilon^{\delta}_{n+P}$. Therefore,

$$\int_{R_0-\gamma\delta}^{R_0+\gamma\delta}\int_{\frac{k\delta}{R_0}}^{\frac{(k+1)\delta}{R_0}}\left|u^{\delta}(r,\theta)-\sum_{k=0}^{n+P}\delta^k U_k(\tfrac{r-R_0}{\delta},\tfrac{R_0\theta}{\delta},\theta)\right|^2 r\; dr\; d\theta \leq C\|\varepsilon^{\delta}_{n+P}\|^2_{H^1(\Omega)}\leq \eta^{2(n+P-1)}$$

and

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|u^{\delta}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})-\sum_{k=0}^{n+P}\delta^k U_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right|^2 dS\; d\mathcal{V} \leq C\eta^{2(n+P-2)}.$$

Choosing $P=4$, and $\eta=\delta^{\frac{n+1}{n+2}}$ gives

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|u^{\delta}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})-\sum_{k=0}^{n+P}\delta^k U_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right|^2 dS\; d\mathcal{V} \leq C\delta^{2(n+1)}. \tag{105}$$

Summing (104) and (105) gives the desired estimate

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|u^{\delta}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})-\sum_{k=0}^{n}\delta^k U_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right|^2 dS\; d\mathcal{V} \leq C\delta^{2(n+1)}.$$

### A.5.2 $\left(\int_{-\gamma}^{\gamma}\int_{\mathbf{j}}^{\mathbf{j+1}}\left|\frac{\partial}{\partial\mathcal{V}}\left(\mathbf{u}^{\delta}(\mathbf{R_0}+\delta\mathcal{V},\tfrac{\mathbf{S}\delta}{\mathbf{R_0}})-\sum_{\mathbf{k=0}}^{\mathbf{n}}\delta^{\mathbf{k}}\mathbf{U_k}(\mathcal{V},\mathbf{S},\tfrac{\mathbf{S}\delta}{\mathbf{R_0}})\right)\right|^{\mathbf{2}}\mathbf{dS}\;\mathbf{d}\mathcal{V}\right)^{\frac{1}{2}}\leq \mathbf{C}\delta^{\mathbf{n+1}}$

Let us define $B$ by

$$B=\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial}{\partial\mathcal{V}}\left(u^{\delta}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})-\sum_{k=0}^{n}\delta^k U_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right)\right|^2 dS\; d\mathcal{V}.$$

Using the triangular inequality, $\forall P\in\mathbb{N}$,

$$B\leq C\left(\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial}{\partial\mathcal{V}}\left(u^{\delta}(R_0+\delta\mathcal{V},\tfrac{S\delta}{R_0})-\sum_{k=0}^{n+P}\delta^k U_k(\mathcal{V},S,\tfrac{S\delta}{R_0})\right)\right|^2 dS\; d\mathcal{V}+\sum_{k=n+1}^{n+P}\delta^{2k}\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial U_k(\mathcal{V},S,\frac{S\delta}{R_0})}{\partial\mathcal{V}}\right|^2 dS\; d\mathcal{V}\right)$$

It is clear that

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial U_k(\mathcal{V},S,\frac{S\delta}{R_0})}{\partial\mathcal{V}}\right|^2 dS \leq C.$$

Consequently,

$$\sum_{k=n+1}^{n+P}\delta^{2k}\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial U_k(\mathcal{V},S,\frac{S\delta}{R_0})}{\partial\mathcal{V}}\right|^2 dS\; d\mathcal{V}\leq C\delta^{2(n+1)}. \tag{106}$$

Moreover,applying the change of scale $\mathcal{V} = \dfrac{r-R_0}{\delta}$, $S = \dfrac{\theta R_0}{\delta}$

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial}{\partial \mathcal{V}}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})}-\sum_{k=0}^{n+P}\delta^k U_{k}{}_{(\mathcal{V},S,\frac{S\delta}{R_0})}\right)\right|^2 dS\ d\mathcal{V}$$

$$\leq \frac{C\delta^2}{\delta^2}\int_{R_0-\gamma\delta}^{R_0+\gamma\delta}\int_{\frac{k\delta}{R_0}}^{\frac{(k+1)\delta}{R_0}}\left|\frac{\partial}{\partial r}\left(u^{\delta}{}_{(r,\theta)}-\sum_{k=0}^{n+P}\delta^k U_{k}{}_{(\frac{r-R_0}{\delta},\frac{R_0\theta}{\delta},\theta)}\right)\right|^2\ r\ dr\ d\theta.$$

Since for $\delta$ small enough,in $]R_0-\gamma\delta, R_0+\gamma\delta[\times]0,2\pi[)$, $u^{\delta}{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})}-\sum_{k=0}^{n+P}\delta^k U_{k}{}_{(\mathcal{V},S,\frac{S\delta}{R_0})}=\varepsilon^{\delta}_{n+P}$,

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial}{\partial \mathcal{V}}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})}-\sum_{k=0}^{n+P}\delta^k U_{k}{}_{(\mathcal{V},S,\frac{S\delta}{R_0})}\right)\right|^2 dS\ d\mathcal{V}\leq C\eta^{2(n+P-1)}.$$

Choosing $P=3$, and $\eta=\delta^{\frac{n+1}{n+2}}$ gives

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial}{\partial \mathcal{V}}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})}-\sum_{k=0}^{n+P}\delta^k U_{k}{}_{(\mathcal{V},S,\frac{S\delta}{R_0})}\right)\right|^2 dS\ d\mathcal{V}.\leq C\delta^{2(n+1)} \tag{107}$$

Summing (106) and (107) gives the desired result.

### A.5.3 $\left(\int_{-\gamma}^{\gamma}\int_{\mathbf{j}}^{\mathbf{j+1}}\left|\frac{\partial}{\partial \mathbf{S}}\left(\mathbf{u}^{\delta}{}_{(\mathbf{R_0}+\delta\mathcal{V},\frac{\mathbf{S}\delta}{\mathbf{R_0}})}-\sum_{\mathbf{k=0}}^{\mathbf{n}}\delta^{\mathbf{k}}\mathbf{U}_{\mathbf{k}}{}_{(\mathcal{V},\mathbf{S},\frac{\mathbf{S}\delta}{\mathbf{R_0}})}\right)\right|^{\mathbf{2}}\mathbf{dS}\ \mathbf{d}\mathcal{V}\right)^{\frac{1}{2}}\leq \mathbf{C}\delta^{\mathbf{n+1}}$

Let $D$ be defined by

$$D=\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial}{\partial S}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})}-\sum_{k=0}^{n}\delta^k U_{k}{}_{(\mathcal{V},S,\frac{S\delta}{R_0})}\right)\right|^2 dS\ d\mathcal{V}.$$

By the triangular inequality, and using that $\dfrac{\partial}{\partial S}\left(U_k(\mathcal{V},S,\dfrac{\delta S}{R_0})\right)=\dfrac{\partial U_k}{\partial S}(\mathcal{V},S,\dfrac{\delta S}{R_0})+\dfrac{\delta}{R_0}\dfrac{\partial U_k}{\partial \theta}(\mathcal{V},S,\dfrac{\delta S}{R_0})$,

$$\begin{aligned} D\ \ \leq\ \ & \int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial u^{\delta}}{\partial S}{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})}-\sum_{k=0}^{n+P}\delta^k\left(\frac{\partial U_k}{\partial S}(\mathcal{V},S,\frac{\delta S}{R_0})+\frac{\delta}{R_0}\frac{\partial U_k}{\partial \theta}(\mathcal{V},S,\frac{\delta S}{R_0})\right)\right|^2 dS\ d\mathcal{V} \\ & +\sum_{k=n+1}^{n+P}\delta^{2k}\underbrace{\int_{-\gamma}^{\gamma}\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial U_k}{\partial S}{}_{(\mathcal{V},S,\frac{\delta S}{R_0})}\right|^2 dS\ d\mathcal{V}}_{\leq C}+\sum_{k=n+1}^{n+P}\delta^{2k+2}\underbrace{\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial U_k}{\partial \theta}{}_{(\mathcal{V},S,\frac{\delta S}{R_0})}\right|^2 dS\ d\mathcal{V}}_{\leq C}. \end{aligned}$$

Moreover, since $\frac{\partial}{\partial S}=\frac{\delta}{R_0}\frac{\partial}{\partial\theta}$

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial u^{\delta}}{\partial S}{}_{(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})}-\sum_{k=0}^{n+P}\delta^k\left(\frac{\partial U_k}{\partial S}(\mathcal{V},S,\frac{\delta S}{R_0})+\frac{\delta}{R_0}\frac{\partial U_k}{\partial \theta}(\mathcal{V},S,\frac{\delta S}{R_0})\right)\right|^2 dS\ d\mathcal{V}$$

$$\begin{aligned} &\leq C\int_{R_0-\gamma\delta}^{R_0+\gamma\delta}\int_{\frac{k\delta}{R_0}}^{\frac{(k+1)\delta}{R_0}}\left|\frac{du^{\delta}}{d\theta}{}_{(r,\theta)}-\sum_{k=0}^{n+P}\delta^k\frac{\partial U_k}{\partial\theta}{}_{(\frac{r-R_0}{\delta},\frac{R_0\theta}{\delta},\theta)}\right|^2\ r\ dr\ d\theta, \\ &\leq C\|\varepsilon^{\delta}_{n+P}\|^2_{H^1(\Omega)}, \\ &\leq C\eta^{2(n+P-1)}. \end{aligned}$$

Choosing $P=3$, and $\eta=\delta^{\frac{n+1}{n+2}}$ gives

$$\int_{-\gamma}^{\gamma}\int_{j}^{j+1}\left|\frac{\partial u^\delta}{\partial S}(R_0+\delta\mathcal{V},\frac{S\delta}{R_0})-\sum_{k=0}^{n+P}\delta^k\left(\frac{\partial U_k}{\partial S}(\mathcal{V},S,\frac{\delta S}{R_0})+\frac{\delta}{R_0}\frac{\partial U_k}{\partial\theta}(\mathcal{V},S,\frac{\delta S}{R_0})\right)\right|^2 dS\;d\mathcal{V}\le C\delta^{2(n+1)}.$$

Finally,

$$D\le C\delta^{2(n+1)}.$$

**A.5.4** $\left(\int_{-\gamma}^{\gamma}\int_{\mathbf{0}}^{\mathbf{2\pi}}\left|\mathbf{u}^\delta(\mathbf{R_0}+\delta\mathcal{V},\theta)-\sum_{\mathbf{k=0}}^{\mathbf{n}}\delta^{\mathbf{k}}\mathbf{U_k}(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)\right|^{\mathbf{2}}\mathbf{d}\theta\;\mathbf{d}\mathcal{V}\right)^{\frac{1}{2}}\le\mathbf{C}\delta^{\mathbf{n+1}}$

Let

$$E=\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|u^\delta(R_0+\delta\mathcal{V},\theta)-\sum_{k=0}^{n}\delta^kU_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)\right|^2 d\theta\;d\mathcal{V}.$$

Again, by triangular inequality,

$$E\le C\left(\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|u^\delta(R_0+\delta\mathcal{V},\theta)-\sum_{k=0}^{n+P}\delta^kU_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)\right|^2 d\theta\;d\mathcal{V}+\sum_{k=n+1}^{n+P}\delta^{2k}\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|U_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)\right|^2 d\theta\;d\mathcal{V}\right).$$

Applying the change of scale $\mathcal{V}=\frac{r-R_0}{\delta}$, it is easily seen that

$$\begin{aligned}\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|u^\delta(R_0+\delta\mathcal{V},\theta)-\sum_{k=0}^{n+P}\delta^kU_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)\right|^2 d\theta\;d\mathcal{V}&\le\frac{C}{\delta}\int_{0}^{2\pi}\int_{R_0-\gamma\delta}^{R_0+\gamma\delta}\left|u^\delta(r,\theta)-\sum_{k=0}^{n+P}\delta^kU_k(\frac{r-R_0}{\delta},\frac{R_0\theta}{\delta},\theta)\right|^2 rdrd\theta,\\&\le\eta^{2(n+P-1)-1}.\end{aligned}$$

Choosing $P=4$, and $\eta=\delta^{\frac{n+1}{n+\frac{5}{2}}}$ yields

$$\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|u^\delta(R_0+\delta\mathcal{V},\theta)-\sum_{k=0}^{n+P}\delta^kU_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)\right|^2 d\theta\;d\mathcal{V}\;\le C\delta^{2(n+1)}.\tag{108}$$

Moreover,

$$\begin{aligned}\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|U_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)\right|^2 d\theta\;d\mathcal{V}&\le C\sum_{j=0}^{\frac{2\pi R_0}{\delta}}\delta\int_{j}^{j+1}\int_{-\gamma}^{\gamma}\left|U_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)\right|^2 dS\;d\mathcal{V},\\&\le C.\end{aligned}$$

Consequently,

$$\sum_{k=n+1}^{n+P}\delta^{2k}\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|U_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)\right|^2 d\theta\;d\mathcal{V}\le C\delta^{n+1}.\tag{109}$$

It suffices to combine the estimates (108) and (109) to obtain the desired conclusion.

**A.5.5** $$\left(\int_{-\gamma}^{\gamma}\int_{\mathbf{0}}^{\mathbf{2\pi}}\left|\frac{\partial}{\partial\mathcal{V}}\left(\mathbf{u}^{\delta}{}_{(\mathbf{R_0}+\delta\mathcal{V},\theta)}-\sum_{\mathbf{k=0}}^{\mathbf{n}}\delta^{\mathbf{k}}\mathbf{U_k}(\mathcal{V},\tfrac{\mathbb{R}_0\theta}{\delta},\theta)\right)\right|^{\mathbf{2}}\mathbf{d}\theta\,\mathbf{d}\mathcal{V}\right)^{\frac{1}{2}}\leq\mathbf{C}\delta^{\mathbf{n+1}}$$

Let us introduce

$$F=\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|\frac{\partial}{\partial\mathcal{V}}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\theta)}-\sum_{k=0}^{n}\delta^{k}U_k(\mathcal{V},\tfrac{\mathbb{R}_0\theta}{\delta},\theta)\right)\right|^{2}d\theta\,d\mathcal{V}.$$

It is easily seen that,$\forall P\in\mathbb{N}$,

$$F\leq\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|\frac{\partial}{\partial\mathcal{V}}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\theta)}-\sum_{k=0}^{n+P}\delta^{k}U_k(\mathcal{V},\tfrac{\mathbb{R}_0\theta}{\delta},\theta)\right)\right|^{2}d\theta\,d\mathcal{V}+\sum_{k=n+1}^{n+P}\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|\frac{\partial U_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)}{\partial\mathcal{V}}\right|^{2}d\theta\,d\mathcal{V}.$$

Again, applying the change of scale $\mathcal{V}=\frac{r-R_0}{\delta}$,

$$\begin{aligned}&\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|\frac{\partial}{\partial\mathcal{V}}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\theta)}-\sum_{k=0}^{n+P}\delta^{k}U_k(\mathcal{V},\tfrac{\mathbb{R}_0\theta}{\delta},\theta)\right)\right|^{2}d\theta\,d\mathcal{V}\\&\qquad\leq\frac{C\delta^2}{\delta}\int_{0}^{2\pi}\int_{R_0-\gamma\delta}^{R_0+\gamma\delta}\left|\frac{\partial}{\partial r}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\theta)}-\sum_{k=0}^{n+P}\delta^{k}U_k(\mathcal{V},\tfrac{\mathbb{R}_0\theta}{\delta},\theta)\right)\right|^{2}\,r\,dr\,d\theta,\\&\qquad\leq C\delta\|u^{\delta}-\sum_{k=0}^{n+P}\delta^{k}U_k\|^2_{H^1(\Omega)},\\&\qquad\leq C\eta^{2(n+P-1)-1}.\end{aligned}$$

Choosing $P=4$, and $\eta=\delta^{\frac{n+1}{n+\frac{5}{2}}}$ yields

$$\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|\frac{\partial}{\partial\mathcal{V}}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\theta)}-\sum_{k=0}^{n+P}\delta^{k}U_k(\mathcal{V},\tfrac{\mathbb{R}_0\theta}{\delta},\theta)\right)\right|^{2}d\theta\,d\mathcal{V}\leq\delta^{n+1}.\tag{110}$$

Moreover,

$$\begin{aligned}\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|\frac{\partial U_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)}{\partial\mathcal{V}}\right|^{2}d\theta\,d\mathcal{V}\quad&\leq\quad C\sum_{j=0}^{\frac{2\pi R_0}{\delta}}\delta\int_{j}^{j+1}\int_{-\gamma}^{\gamma}\left|\frac{\partial U_k(\mathcal{V},S,\frac{R_0\theta}{\delta})}{\partial\mathcal{V}}\right|^{2}d\mathcal{V}dS,\\&\leq\quad C.\end{aligned}$$

So,

$$\sum_{k=n+1}^{n+P}\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|\frac{\partial U_k(\mathcal{V},\frac{\mathbb{R}_0\theta}{\delta},\theta)}{\partial\mathcal{V}}\right|^{2}d\theta\,d\mathcal{V}\leq C\delta^{2(n+1)}.\tag{111}$$

Adding (110) and (111) establishes the desired conclusion.

Note that we only have

$$\left(\int_{-\gamma}^{\gamma}\int_{0}^{2\pi}\left|\frac{\partial}{\partial\theta}\left(u^{\delta}{}_{(R_0+\delta\mathcal{V},\theta)}-\sum_{k=0}^{n}\delta^{k}U_k(\mathcal{V},\tfrac{\mathbb{R}_0\theta}{\delta},\theta)\right)\right|^{2}d\theta\,d\mathcal{V}\right)^{\frac{1}{2}}\leq C\delta^{n}.$$

## A.6 Proof of Proposition 4.1

**Proposition 4.1.** *There exist* $\delta_0 > 0$, $h_0 > 0$ *and a constant* $C > 0$ *such that*

$$\forall \delta < \delta_0, \forall h < h_0, \qquad \inf_{u_h \in V^h_{\alpha\delta}} \sup_{v_h \in V^h_{\alpha\delta}} \frac{\left|\breve{a}^\delta(u_h, v_h)\right|}{\|u_h\|_{\breve{V}_{\alpha\delta}} \|v_h\|_{\breve{V}_{\alpha\delta}}} \;\geq\; C \tag{87}$$

*Proof.* The proof is done by contradiction. Suppose that ((87)) is false. Then, $\forall \delta_0 > 0, \forall h_0 > 0, \forall C > 0$, there exist $\delta < \delta_0$, $h < h_0$ and $u^\delta_h \in V^h_{\alpha\delta}$ such that

$$\|u^\delta_h\|_{\breve{V}_{\alpha\delta}} > \frac{1}{C} \sup_{v_h \in V^h_{\alpha\delta}} \frac{\left|\breve{a}^\delta(u_h, v_h)\right|}{\|v_h\|_{\breve{V}_{\alpha\delta}}}.$$

Therefore, there exist 3 sequences $(\delta_n)_{n\in\mathbb{N}}$, $(h_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ such that

- $\delta_n > 0 \ \forall n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} \delta_n = 0$,
- $h_n > 0 \ \forall n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} h_n = 0$,
- $u_n \in V^{h_n}_{\alpha\delta_n} \ \forall n \in \mathbb{N}$ and

$$\|u_n\|_{\breve{V}_{\alpha\delta_n}} = 1 \quad \forall n \in \mathbb{N}, \quad (a)$$

$$\lim_{n\to+\infty} \sup_{v_n \in V^{h_n}_{\alpha\delta_n}} \frac{\left|\breve{a}^{\delta_n}(u_n, v_n)\right|}{\|v_n\|_{\breve{V}_{\alpha\delta_n}}} = 0. \quad (b)$$

As in the continuous case, we define $\hat{u}_n$

$$\hat{u}_n(\hat{x}) = \begin{cases} u_n \circ F^{\delta+}(\hat{x}) & \text{if } |\hat{x}| < R_0, \\ u_n \circ F^{\delta-}(\hat{x}) & \text{if } |\hat{x}| > R_0. \end{cases}$$

and the bilinear form $\hat{a}^{\delta_n}$

$$\begin{aligned}
\hat{a}^{\delta_n}(\hat{u}_n, \hat{v}) \qquad &:= \breve{a}^{\delta_n}(u_n, v), \\
&:= \int_{\Omega^+} (DF^{\delta+}(\hat{x})^{-1})(DF^{\delta+}(\hat{x})^{-1*})\mu_\infty \hat{\nabla}\hat{u}_n \cdot \overline{\hat{\nabla}\hat{v}} \ \left|det(DF^{\delta+})\right| + \int_{S_{R_e}} i\omega\mu_\infty \hat{u}_n \bar{\hat{v}} d\sigma \\
&+ \int_{\Omega^-} (DF^{\delta-}(\hat{x})^{-1})(DF^{\delta-}(\hat{x})^{-1*})\mu_\infty \hat{\nabla}\hat{u}_n \cdot \overline{\hat{\nabla}\hat{v}} \ \left|det(DF^{\delta-})\right| \\
&- \int_{\Omega^+} \rho_\infty \omega^2 \hat{u}_n \bar{\hat{v}} \ \left|det(DF^{\delta+})\right| - \int_{\Omega^-} \rho_\infty \omega^2 \hat{u}_n \bar{\hat{v}} \ \left|det(DF^{\delta-})\right| \\
&- \delta B^\alpha_2 \mu_\infty \int_0^{2\pi} \left(\frac{\partial \hat{u}_n}{\partial\theta}\right)^- \left(\frac{\partial \bar{\hat{v}}}{\partial\theta}\right)^- d\theta + \delta B^\alpha_1 \mu_\infty \int_0^{2\pi} (\hat{u}_n)^- \, (\hat{v})^- \, d\theta \\
&+ \frac{\mu_\infty}{\delta A^\alpha_0} \int_0^{2\pi} [\hat{u}_n][\bar{\hat{v}}] d\theta.
\end{aligned}$$

Using $(a)$ and the uniform continuity of $F^\delta$ it is clear that there exist two positive constants $A$ and $B$ such that

$$0 < A \leq \|\hat{u}_n\|_{\breve{V}_0} \leq B, \quad (a')$$

where the norm on $V_0$ is defined by (68). Moreover, combining the uniform continuity of $F^\delta$ with the assumptions (H3) and (b), we obtain

$$\lim_{n\to+\infty} \sup_{v_n\in V_0^{h_n}} \frac{\left|\hat{a}^{\delta_n}(\hat{u}_n, v_n)\right|}{\|v_n\|_{\breve{V}_0}} = 0. \quad (b')$$

Therefore, there is a sub-sequence (still denoted by $(\hat{u}_n)$) and a function $\hat{u}_0 \in H^1(\Omega^+) \cup H^1(\Omega^-)$ such that:

$$\begin{aligned} &\hat{u}_n \rightharpoonup \hat{u}_0^+ \text{ weakly in } H^1(\Omega^+),\\ &\hat{u}_n \rightharpoonup \hat{u}_0^- \text{ weakly in } H^1(\Omega^-),\\ &\hat{u}_n \rightharpoonup \hat{u}_0^\pm \text{ weakly in } H^{1/2}(S_{R_0^\pm}). \end{aligned}$$

Moreover, it is clear that

$$\int_0^{2\pi} |[\hat{u}_0]|^2 \, d\theta = 0.$$

Let $v \in \breve{V}_0 \cap H^1(\Omega)$. Using the assumption (H2), it is clear that there exists a sequence $(v_n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N} \quad v_n \in V_0^{h_n} \cap H^1(\Omega) \quad \text{and} \qquad \lim_{n\to\infty} \|v - v_n\|_{H^1(\Omega^+)} + \|v - v_n\|_{H^1(\Omega^-)} + \|(v - v_n)^-\|_{H^1(]0,2\pi[)} = 0.$$

Moreover, there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$\forall n > N \quad \|v_n\|_{\breve{V}_0} \le C\|v\|_{\breve{V}_0}. \tag{112}$$

In addition,

$$\hat{a}^{\delta_n}(\hat{u}_n, v) = \hat{a}^{\delta_n}(\hat{u}_n, v - v_n) + \hat{a}^{\delta_n}(\hat{u}_n, v_n).$$

It is clear that

$$\lim_{n\to+\infty} \hat{a}^{\delta_n}(\hat{u}_n, v) = \int_\Omega \mu_\infty \nabla\hat{u}_0 \cdot \nabla\bar{v} - \omega^2\rho_\infty\hat{u}_0\bar{v} + \int_{S_{R_e}} i\mu_\infty\omega\hat{u}_0\bar{v}. \tag{113}$$

Using the assumption (H3) gives

$$\lim_{n\to+\infty} \hat{a}^{\delta_n}(\hat{u}_n, v - v_n) = 0. \tag{114}$$

Moreover, using (112) we obtain

$$\hat{a}^{\delta_n}(\hat{u}_n, v_n) \le C \sup_{w_n\in V_0^{h_n}} \frac{|\hat{a}^{\delta_n}(\hat{u}_n, w_n)|}{\|w_n\|_{\breve{V}_0}} \|v\|_{\breve{V}_0}.$$

Therefore,

$$\lim_{n\to+\infty} \hat{a}^{\delta_n}(\hat{u}_n, v_n) = 0.$$

Combining (114), (113) with (113) yields

$$\forall v \in \breve{V}_0 \cap H^1(\Omega), \quad \int_\Omega \mu_\infty \nabla\hat{u}_0 \cdot \nabla\bar{v} - \omega^2\rho_\infty\hat{u}_0\bar{v} + \int_{S_{R_e}} i\mu_\infty\omega\hat{u}_0\bar{v} = 0.$$

By density of $\breve{V}_0 \cap H^1(\Omega)$ in $H^1(\Omega)$,

$$\forall v \in H^1(\Omega), \quad \int_\Omega \mu_\infty \nabla\hat{u}_0 \cdot \nabla\bar{v} - \omega^2\rho_\infty\hat{u}_0\bar{v} + \int_{S_{R_e}} i\mu_\infty\omega\hat{u}_0\bar{v} = 0.$$

It follows that $\hat{u}_0 = 0$.
We end the proof by proving that $\lim\limits_{n\to+\infty} \|\hat{u}_n\|_{\check{V}_0} = 0$.

$$\|\hat{u}_n\|^2_{\check{V}_0} \leq C\left(|\hat{a}^{\delta_n}(\hat{u}_n, \hat{u}_n)| + \|\hat{u}_n\|_{L^2(\Omega^+)} + \|\hat{u}_n\|_{L^2(\Omega^-)} + \|\sqrt{\delta}(\hat{u}_n)^-\|_{L^2(]0,2\pi[)}\right)$$

Since the right side hand of the previous inequality tends to $0$, $\|\hat{u}_n\|_{\check{V}_0}$ tends to $0$. It contradicts the assumption (a') and proves the stability result (87). □

# B The Whole Expansion

In this part we prove the proposition 2.8: for each $n \in \mathbb{N}$, $u_n$ and $U_n$ exist and are unique. To do that, we extend the results of the subsections 2.2.2 and 2.2.3 to each $n$ in $\mathbb{N}$.

- In a first step, we write the matching conditions in a new form. This new form is optimal in the sense that we only match the coefficients that have to be matched : the redundant conditions are eliminated.
- In a second step (which is completely independent of the first one) we introduce two families of functions $(V_n^k)_{(n\in\mathbb{N},k\leq n)}$ and $(W_n^k)_{(n\in\mathbb{N},k\leq n)}$: as for $U_0$, $U_1$ and $U_2$, these functions allow us to separate microscopic variables from macroscopic one $\theta$ at the time of the building of $U_n$.
- Finally, using the results of the two previous steps, we are able to build recursively $u_n$ and $U_n$.

## B.1 A New Version of the Matching Conditions

This part is directly inspired by the second chapter of the PhD thesis of X. Claeys ([11]). Our objective is to obtain an optimal version of the matching conditions: we would like to eliminate redundant data. The following figures illustrates the setting of the problems.

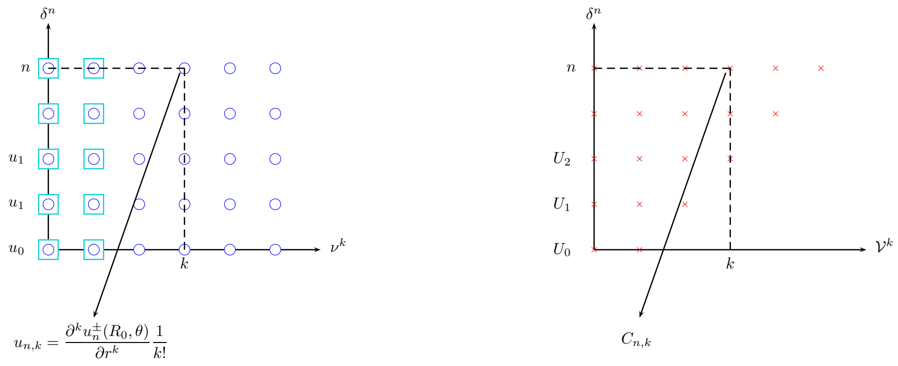



Figure 15: Schematic figure of far fields (left) and near field (right) expansion in the overlapping zones

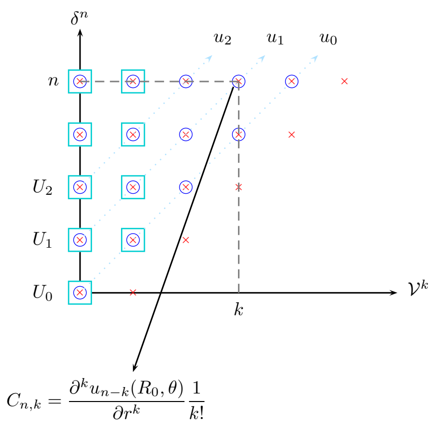



Figure 16: Schematic figure of the matching conditions

- In the left side of the figure 15, the expansion of far field in the overlapping zone is represented in the plan $(\nu^k, \delta^n)$:

$$u^\delta = \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \delta^n \nu^k u_{n,k} = \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \delta^n \nu^k \frac{\partial^k u_n^\pm(R_0,\theta)}{\partial r^k}.$$

  In the line $n$, you can see the different terms of the expansion of $u_n$ according to $\nu^k$.
  Since $u_n$ is solution of the homogeneous Helmholtz equation (second order partial differential equation), we will see that the expansion of $u_n$ according to $\nu$ only depends on the two first terms $u_n^\pm(R_0,\theta)$ and $\dfrac{\partial u_n^\pm}{\partial r}(R_0,\theta)$. Consequently there are only two coefficients that can be matched. That is why the two first terms of each line are twice surrounded.

- In the right side of the figure 15, you can see the expansion of the near field in the overlapping zone represented in the plan $(\mathcal{V}^k, \delta^n)$:

$$u^\delta = \sum_{n\in\mathbb{N}}\sum_{k=0}^{n+1} \delta^n \mathcal{V}^k C_{n,k}^\pm.$$

  In the line $n$, you can see the different terms of the expansion of $U_n$ according to $\mathcal{V}^k$.

- In the figure 16, the far field and near field expansion are both represented in the plan $(\mathcal{V}^k, \delta^n)$. Note that the far field expansion in the plan $(\mathcal{V}^k, \delta^n)$ is obtained by a rotation of $\dfrac{\pi}{4}$ of the far field expansion representation in the plan $(\nu^k, \delta^n)$.

In the figure 16, we immediately recognize the first version of the matching conditions

$$\forall n \in \mathbb{N}, \quad \begin{cases} C_{n,k} = \dfrac{1}{k!}\dfrac{\partial^k u_{n-k}^\pm}{\partial r^k} & \text{if } k \le n, \\ 0 & \text{if } k = n. \end{cases}$$

However, in the figure 16, the only terms that can be matched are the twice surrounded terms: these terms correspond to the fundamental matching conditions

$$\forall n \in \mathbb{N}, \quad C_{n,0}^\pm = u_n^\pm \qquad \text{and} \quad C_{n,1}^\pm = \frac{\partial u_{n-1}^\pm}{\partial r}. \tag{115}$$

That means that the others matching conditions ($C_{n,k} = \dfrac{1}{k!}\dfrac{\partial^k u_{n-k}}{\partial r}$ $k \ge 2$) have to be redundant (if they are not, we cannot match the far field expansion and near field expansion): indeed, assume that $(U_k)_{k\le n-1}$ and $(u_k)_{k\le n-2}$ are known. Then, if $k \ge 2$, $\dfrac{1}{k!}\dfrac{\partial^k u_{n-k}}{\partial r}$ is always completely determined and consequently $C_{n,k}$ is also completely determined.

The purpose of this part is to understand why, for a fixed $n$, theses redundant conditions hold if the fundamental matching conditions (115) hold for $k \le n-1$.

To do that, we are going to study the operator $r^2(\mu_\infty\Delta + \omega^2\rho_\infty)$ according to $\nu$: indeed, the matching of the expansion hold because the far field expansion and the near field expansion are solutions of the same homogeneous Helmholtz equation.

**Decomposition of the operator $r^2(\mu_\infty\Delta + \omega^2\rho_\infty)$ in term of $\nu$**

Using the expression of the Laplace operator in the polar coordinates, we can decompose the operator $r^2(\mu_\infty\Delta + \omega^2\rho_\infty)$ according to $\nu$

$$\mathcal{A}u = r^2(\mu_\infty\Delta + \omega^2\rho_\infty)u = \frac{1}{\nu^2}\sum_{j=0}^{4}\nu^j\mathcal{A}_j^f u, \tag{116}$$

where $A_j^f(\nu\frac{\partial}{\partial\nu}, \frac{\partial}{\partial\theta}), (j = 0\cdots 4)$, are given by

$$\begin{aligned}
&\mathcal{A}_0^f u = \left(R_0^2\mu_\infty\ \nu^2\ \frac{\partial^2}{\partial\nu^2},\right)u = R_0^2\mu_\infty\ \left((\nu\frac{\partial}{\partial\nu})^2 - \nu\frac{\partial}{\partial\nu}\right)u,\\
&\mathcal{A}_1^f = R_0\mu_\infty\ (\nu^2\frac{\partial^2}{\partial\nu^2} + (\nu\frac{\partial}{\partial\nu})^2)u = R_0\mu_\infty\ \left(2(\nu\frac{\partial}{\partial\nu})^2 - \nu\frac{\partial}{\partial\nu}\right)u,\\
&\mathcal{A}_2^f u = \left(\mu_\infty\ \left((\nu\frac{\partial}{\partial\nu})^2 + \frac{\partial^2}{\partial\theta^2}\right) +\ \rho_\infty\omega^2R_0^2\right)u,\\
&\mathcal{A}_3^f u = 2\omega^2\rho_\infty R_0 u,\\
&\mathcal{A}_4^f u = \omega^2\rho_\infty u.
\end{aligned}$$

The operators $\mathcal{A}_j^f$ has the following interesting properties:

**Proposition B.1.** *The operators $\mathcal{A}_j^f$ are homogeneous for two reasons:*

- *They are unchanged by the scale change $\mathcal{V} = \dfrac{\nu}{\delta}$.*
- *If $u(\nu,\theta) = \nu^k u_k(\theta)$, there exists a function $\tilde{u}_k$ which only depends on $\theta$ such that $\mathcal{A}_j^f u = \nu^k\tilde{u}_k(\theta)$.*

**Remark B.2.** *We can compare these operators with the operators (of near field) $\mathcal{A}_{j,l}^f$ defined in* (92)*:*

- $\mathcal{A}_{2,l}^f = \mathcal{A}_2^f,\quad \mathcal{A}_{3,l}^f = \mathcal{V}\mathcal{A}_3^f\ ,$ *and* $\mathcal{A}_{4,l}^f = \mathcal{V}^2\mathcal{A}_4^f.$
- *It is also possible to compare $\mathcal{A}_{0,0}^f$ and $\mathcal{A}_{1,0}^f$ with $\mathcal{A}_0^f$ and $\mathcal{A}_1^f$.*

$$\mathcal{A}_{0,0}^f = \frac{1}{\mathcal{V}^2}\mathcal{A}_0^f \quad \textit{and} \quad \mathcal{A}_{1,0}^f = \frac{1}{\mathcal{V}}\mathcal{A}_1^f.$$

**Modal expansion of the far field terms**

We remind that in the vicinity of the interface $S_{R_0}$, $u_n$ is regular: its expansion according to $\nu$ is given by

$$u_n^\pm(r,\theta) = \sum_{k\in\mathbb{N}}\nu^k u_{n,k}^\pm(\theta) \quad \text{where} \quad u_{n,k}^\pm(\theta) = \frac{1}{k!}\frac{\partial^k u_n^\pm(R_0,\theta)}{\partial r^k}. \tag{117}$$

Moreover $u_n$ satisfies the homogeneous Helmholtz equation.

The goal of this paragraph is to prove that the expansion of $u_n$ only depends on the two Cauchy data $u_n^\pm(R_0,\theta)$ and $\dfrac{\partial u_n^\pm}{\partial r}(R_0,\theta)$.

To shorten notation, we define the space of formal series $V_0$, a subset of $V_0$ called $V_0^k$ and two linear forms $l_0$ and $l_1$ on $V_0$:

$$V_0 = \{v(\nu,\theta) \text{ such that } v(\nu,\theta) = \sum_{k\in\mathbb{N}} \nu^k v_k(\theta)\},$$

$$V_0^k = \left\{v / v(\nu,\theta) = \nu^k v_k(\theta), v_k(\theta) \in \mathcal{C}^\infty([0,2\pi])\right\},$$

$$l_0 : \begin{cases} V_0 \to \mathcal{C}^\infty([0,2\pi]), \\ v = \displaystyle\sum_{p\in N} \nu^p v_p(\theta) \mapsto l_0(v) = v_0(\theta), \end{cases}$$

$$l_1 : \begin{cases} V_0 \to \mathcal{C}^\infty([0,2\pi]), \\ v = \displaystyle\sum_{p\in N} \nu^p v_p(\theta) \mapsto l_1(v) = v_1(\theta). \end{cases}$$

For any formal series $v \in V$, $l_0$ (respectively $l_1$) associated it its coefficient of degree $0$ (resp. $1$).

The definition of the convergence of the formal series is the following one:

$$v(\nu,\theta) = \sum_{k\in\mathbb{N}} \nu^k v_k(\theta) \quad \Leftrightarrow \quad \exists \nu_0 > 0,\ \forall \nu \geq \nu_0, \quad |v(\nu,\theta) - \sum_{k=0}^{n} \nu^k v_k(\theta)| \leq C_n |\nu|^{n+1}\ . \tag{118}$$

Let us introduce $u$, such that $u \in V$ and $u$ is solution of the homogeneous Helmholtz equation in the vicinity of $S_{R_0}$:

$$u = \sum_{k\in\mathbb{N}} \nu^k u_k(\theta) \quad \text{and} \quad \mathcal{A}u = 0.$$

Inserting the formal series in the decomposition of the operator $\mathcal{A}$ (116) yields

$$\sum_{k=0}^{4} \sum_{j\in\mathbb{N}} \nu^k \mathcal{A}_k^f \left(\nu^j u_j\right) = 0,$$

which after rearrangements gives

$$\sum_{k\in\mathbb{N}} \nu^k \underbrace{\left(\sum_{j=k-4}^{k} \nu^{-j} \mathcal{A}_{k-j}^f(\nu^j u_j(\theta))\right)}_{\text{independent of } \nu} = 0.$$

Using the homogeneity of the operators $\mathcal{A}_j^f$, it is clear that $\nu^{-j}\mathcal{A}_{k-j}^f(\nu^j u_{n,j}(\theta))$ does not depend on $\nu$. Separating the powers of $\nu$ gives

$$\forall k \in \mathbb{N}, \sum_{j=k-4}^{k} \nu^{-j} \mathcal{A}_{k-j}^f(\nu^j u_j(\theta)) = 0.$$

(we use the convention that $u_{n,j} = 0$ if $j < 0$). Consequently,

$$\mathcal{A}_0^f\left(\nu^k u_k\right) = -\nu^k \left(\sum_{j=k-4}^{k-1} \underbrace{\nu^{-j}\left(\mathcal{A}_{k-j}^f(\nu^j u_j)\right)}_{\text{independent of } \nu}\right). \tag{119}$$

Therefore, if $\mathcal{A}_0^f$ is bijective on the space $V_0^k$, $u_k$ is uniquely determined and only depends on $u_j$ with $j < k$. We need the following obvious property of $\mathcal{A}_0^f$:

**Proposition B.3.**

- $\mathcal{A}_0^f$ *is an isomorphism from* $V_0^k$ *to* $V_0^k$ *if* $p \geq 2$.
- $\mathcal{A}_0^f$ *restricted to* $V_0^0$ *or* $V_0^1$ *is equal to* $0$ *(* $\mathcal{A}_0^f$ *restricted to* $V_0^0$ *or* $V_0^1$ *is neither injective nor surjective).*

Since $\mathcal{A}_1^f$ restricted to $V_0^1$ is equal to $0$,

$$\mathcal{A}_0^0(u_0) = 0 \quad \text{and} \quad \mathcal{A}_0^0(\nu u_1) = 0.$$

Using Proposition B.3, we obtain that $u_0 = l_0(u)$ and $u_1 = l_1(u)$ are undetermined. Then, using again Proposition B.3, the formula (119) entirely defines (by induction) $u_k$ for any $k \geq 2$.

Since $u_k$ only depends on $u_j, (j < k)$, we have proved that $u_k$ only depends on $l_0(u)$ and $l_1(u)$.

To explicitly know the dependence in $l_0(u_n)$ and $l_1(u_n)$ it is useful to consider the operators $s_k^0$, $s_k^1$

$$s_k^0 : \begin{cases} \mathcal{C}^\infty([0,2\pi]) \to \mathcal{C}^\infty([0,2\pi]) \\ s_k^0 = 0, \quad k < 0, \\ s_0^0 = Id, \\ s_1^0 = 0, \\ s_k^0 = \underbrace{\nu^{-k}(\mathcal{A}_0^f)^{-1}\nu^k}_{\text{independent of } \nu} \left( \displaystyle\sum_{j=k-4}^{k-1} \underbrace{\nu^{-j}\mathcal{A}_{k-j}^f(\nu^j s_j^0)}_{\text{independent of } \nu} \right), \quad k \geq 2, \end{cases} \tag{120}$$

$$s_k^1 : \begin{cases} \mathcal{C}^\infty([0,2\pi]) \to \mathcal{C}^\infty([0,2\pi]), \\ s_k^1 = 0, k \leq 0, \\ s_1^1 = Id, \\ s_k^1 = \underbrace{\nu^{-k}(\mathcal{A}_0^f)^{-1}\nu^k}_{\text{independent of } \nu} \left( \displaystyle\sum_{j=k-4}^{k-1} \underbrace{\nu^{-j}\mathcal{A}_{k-j}^f(\nu^j s_j^1)}_{\text{independent of } \nu} \right), \quad k \geq 2, \end{cases} \tag{121}$$

and the operators $s^0$, $s^1$

$$\begin{cases} s^0 : \mathcal{C}^\infty([0,2\pi]) \to V_0 \\ \forall a \in \mathcal{C}^\infty([0,2\pi]) \;\; s^0[a] = \displaystyle\sum_{k \in N} \nu^k s_k^0[a], \end{cases} \tag{122}$$

$$\begin{cases} s^1 : \mathcal{C}^\infty([0,2\pi]) \to V_0 \\ \forall a \in \mathcal{C}^\infty([0,2\pi]) \;\; s^1[a] = \displaystyle\sum_{k \in \mathbb{N}^*} \nu^k s_k^1[a]. \end{cases} \tag{123}$$

Finally we can prove the following result:

**Proposition B.4.** *In the vicinity of* $S_{R_0}$*, the expansion of* $u$ *is given by*

$$u^\pm(r,\theta) = s^0[l^0(u^\pm)] + s^1[l^1(u^\pm)]. \tag{124}$$

*Proof.* To prove (124), we prove by induction that

$$u_k(\theta)^{\pm} = s_k^0(l^0(u^{\pm})) + s_k^1(l^1(u^{\pm})) \quad \forall k. \in \mathbb{N} \tag{125}$$

The initialization is immediate. We assume that for any $j \leq k$ (125) holds.
By (119), we know that

$$\mathcal{A}_0^f\left(\nu^{k+1} u_{k+1}\right) = -\nu^{k+1}\left(\sum_{j=k-3}^{k+1} \nu^{-j}\left(\mathcal{A}_{k+1-j}^f(\nu^j u_j)\right)\right).$$

By assumption, we can replace $u_j$ by $s_j^0(l^0(u^{\pm})) + s_j^1(l^0(u^{\pm})))$. It follows that

$$\mathcal{A}_0^f\left(\nu^{k+1} u_{k+1}\right) = -\nu^{k+1}\left(\sum_{j=k-3}^{k+1} \nu^{-j}\left(\mathcal{A}_{k+1-j}^f(\nu^j s_j^0(l^0(u^{\pm})))\right)\right)$$
$$-\nu^{k+1}\left(\sum_{j=k-3}^{k+1} \nu^{-j}\left(\mathcal{A}_{k+1-j}^f(\nu^j s_j^1(l^1(u^{\pm})))\right)\right).$$

This previous formula exactly proves that (125) holds for $k+1$

$$u_{k+1}^{\pm}(\theta) = s_{k+1}^0(l^0(u^{\pm})) + s_{k+1}^1(l^1(u^{\pm})).$$

□

Consequently the expansion of $u_n$ according to $\nu$ in the overlapping zones can be written as:

$$\boxed{u_n = u_n^{\pm}(r,\theta) = s^0[l^0(u_n^{\pm})] + s^1[l^1(u_n^{\pm})] = s^0[u_n^{\pm}(R_0,\theta)] + s^1[\frac{\partial u_n^{\pm}(R_0,\theta)}{\partial r}].} \tag{126}$$

which means in particular that

$$\frac{1}{k!}\frac{\partial^k u_n^{\pm}(R_0,\theta)}{\partial r^k} = s_k^0(u_n^{\pm}(R_0,\theta)) + s_k^1(\frac{\partial u_n^{\pm}(R_0,\theta)}{\partial r}) \quad \forall\, k \geq 2.$$

In the formula (126), it is easily seen that, for any $n \in \mathbb{N}$, the only coefficients that can be matched are $l_0(u_n)$ and $l_1(u_n)$.

**Modal expansion of the near field terms**

We remind that we have prove that $U_n$ has the following behaviour for large $\mathcal{V}$((20) and (21))

for $\mathcal{V} \geq \frac{1}{2}$

$$U_n(\mathcal{V}, S, \theta) = \sum_{k=0}^{n+1} C_{n,k}^{+}(\theta)\mathcal{V}^k + o(\mathcal{V}^{-\infty}),$$

for $\mathcal{V} \leq -\frac{1}{2}$

$$U_n(\mathcal{V}, S, \theta) = \sum_{k=0}^{n+1} C_{n,k}^{-}(\theta)\mathcal{V}^k + o(\mathcal{V}^{-\infty}).$$

Using the new expansion of $u_n$ (126) and looking at the figure 16 we can visualize the following proposition:

**Proposition B.5.**

$$\boxed{\begin{array}{ll} C^{\pm}_{n,0} = s^0_0(C^{\pm}_{n,0}) & (C^{\pm}_{n,0} \textit{ undetermined}), \\ C^{\pm}_{n,1} = s^1_1(C^{\pm}_{n,1}) & (C^{\pm}_{n,1} \textit{ undetermined}), \\ C^{\pm}_{n,k} = s^0_k(C_{n-k,0}) + s^1_k(C_{n-k+1,1}). & \end{array}} \tag{127}$$

*Proof.* The proof is done by induction on $n$. The initialization is trivial. We know that $U_{n+1}$ verifies

$$\mathcal{A}_0 U_{n+1} = -\sum_{k=0}^{4} \mathcal{A}_k U_{n+1-k}.$$

Consequently the constant (in S) Fourier coefficient $(U_{n+1})^+_0$ satisfies

$$\mathcal{A}^f_{0,0}((U_{n+1})^+_0) = -\sum_{k=0}^{4} \mathcal{A}^f_{k,0}(U_{n+1-k})^+_0.$$

Using the remark B.2, we can replace $\mathcal{A}^f_{k,0}$ by $\mathcal{V}^k \mathcal{A}^f_k$:

$$\begin{aligned} \sum_{k=0}^{n+2} \mathcal{A}^f_0(C^+_{n+1,k}(\theta)\mathcal{V}^k) &= \sum_{j=1}^{4} \sum_{k=0}^{n+2-j} \mathcal{V}^j \mathcal{A}^f_j(\mathcal{V}^k(s^0_k(C_{n-j-k+1,0})) + s^1_k(C_{n-j-k+2,1})), \\ &= \sum_{j=1}^{4} \sum_{k=0}^{n+2-j} \mathcal{V}^{j+k} \left(\mathcal{V}^{-k} \mathcal{A}^f_j(\mathcal{V}^k(s^0_k(C_{n-j-k+1,0})) + s^1_k(C_{n-j-k+2,1}))\right), \\ &= \sum_{j=0}^{n+2} \mathcal{V}^j \left( \sum_{k=j-4}^{j-1} \underbrace{\mathcal{V}^{-k} \mathcal{A}^f_{j-k}(\mathcal{V}^k(s^0_k(C_{n-j+1,0})) + s^1_k(C_{n-j+2,1}))}_{\text{independent of} \mathcal{V}^k} \right). \end{aligned} \tag{128}$$

We can also remark that

$$\sum_{k=0}^{n+1} \mathcal{A}^f_0(C^+_{n+1,k}(\theta)\mathcal{V}^k) = \sum_{j=0}^{n+1} \mathcal{V}^{-j} \left(\mathcal{V}^{-j} \mathcal{A}^f_0(C^+_{n+1,j}(\theta)\mathcal{V}^j)\right).$$

Since the operators $\mathcal{A}^f_j$ are homogeneous, we can formally identify the powers of $\mathcal{V}$ in (128).

- If $j = 0$, $\mathcal{A}^f_0(C^+_{n+1,0}) = 0$. Consequently $C_{n+1,0}$ is undetermined and (127) holds.

- If $j = 1$, we remark that $\mathcal{A}^f_1(C_{n+1,0}) = 0$ and $s^1_0 = 0$. It follows that

  $$\mathcal{V}^{-1} \mathcal{A}^f_0(\mathcal{V} C_{n+1,1}) = 0.$$

  Consequently $C_{n+1,1}$ is undetermined and (128) is proved in this case.

- if $2 \le j \le n+1$, the identification yields,

  $$C^+_{n+1,j}(\theta) = \mathcal{V}^{-j} \left( (\mathcal{A}^f_0)^{-1} \left( \mathcal{V}^j \sum_{k=j-4}^{j-1} \mathcal{V}^{-k} \mathcal{A}^f_{j-k}(\mathcal{V}^k(s^0_k(C_{n+1-j,0})) + s^1_k(C_{n-j+2,1})) \right) \right).$$

  It exactly proves that

  $$C^+_{n+1,j}(\theta) = s^0_j(C_{0,n+1-j}) + s^1_j(C_{n+1-j+1,1}).$$

- Finally if $j = n+2$, we remark that $s_k^0(L_0(U_{-1})) = 0$. So

$$C^+_{n+1,n+2}(\theta) = \mathcal{V}^{-(n+1)}\left((\mathcal{A}_0^f)^{-1}\left(\mathcal{V}^j \sum_{k=n-2}^{n+1} \mathcal{V}^{-k}\mathcal{A}^f_{n+2-k}(\mathcal{V}^k s_k^1(C_{0,1}))\right)\right).$$

Therefore,

$$C^+_{n+1,n+2}(\theta) = s^1_{n+1}(C_{0,1}).$$

□

**Remark B.6.**

- *The proposition B.5 means that if we know $C_{k,0}$ and $C_{k,1}$ for any $k < n$, we know entirely the behaviour of $U_k$ for any $k < n$ and we also know $C_{n+1,k}$ for any $k \geq 2$. That is why the only two coefficients to match at each step are $C^{\pm}_{n,0}$ and $C^{\pm}_{n,1}$.*
- *The functions $s^0_k$ and $s^1_k$ appear as in the far field term expansion but there is now also a shift in $n$. It is of course reasonable to also obtain the same recursive functions as in the far field. The shift in $n$ can be visualize on the figure 16 and directly follows from the scale change: indeed,*

$$r^2(\mu_\infty\Delta + \omega^2\rho_\infty)U(\frac{\nu}{\delta}) = \frac{1}{\delta^2}\left(\sum_{j=0}^{4}\delta^j\mathcal{V}^j(\mathcal{A} - j^f U)\right)_{|\mathcal{V}=\frac{\nu}{\delta}}.$$

**A new version of the matching conditions**
We are now in a position to derive a new form of the matching conditions: it is now clear that the coefficients which have to be match at each step are $C^{\pm}_{n,0}, C^{\pm}_{n,1}$ for the near fields terms and $l_0(u_n)$ and $l_1(u_n)$ for the near fields. But,

$$\begin{aligned}
\sum_{n\in\mathbb{N}}\delta^n U_n(\theta, \frac{R_0\theta}{\delta}, \frac{\nu}{\delta}) &= \sum_{n\in\mathbb{N}}\delta^n\sum_{k=0}^{n+1}\frac{\nu^k}{\delta^k}\left(s^0_k(C_{n-k,0}) + s^k_1(C_{n-k+1,1})\right) + O(\delta^{+\infty}),\\
&= \sum_{n\geq 1}\sum_{k\in\mathbb{N}}\delta^n\nu^k\left(s^0_k(C_{n,0}) + s^k_1(C_{n+1,1})\right) + O(\delta^{+\infty}),
\end{aligned}$$

and

$$\sum_{n\in\mathbb{N}}\delta^n u_n = \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}\delta^n\nu^k\left(s^0_k(u_n) + s^k_1(\frac{\partial u_{n-1}}{\partial r})\right).$$

Consequently, we have proved these following optimal matching conditions:

**Proposition B.7.**
*The matching conditions* (24) *are equivalent to the following ones.*

$$\boxed{C^{\pm}_{n,0} = u_n(R_0^{\pm},\theta), \qquad C^{\pm}_{n,1} = \frac{\partial u_{n-1}(R_0^{\pm},\theta)}{\partial r}.} \tag{129}$$

## B.2 The Families $(V_n^k)$ and $(W_n^k)$

In the subsection 2.2.2, we have build step by step $U_0$, $U_1$ and $U_2$. In each step it was possible to separate the variable $\theta$ from the variables $\mathcal{V}$ and $S$ introducing functions which do not depend on $\theta$: for instance, to build $U_1$ we have introduce $V_1^0$, $V_1^1$ and $W_0^0$.

In the same way, the functions $(W_{n-1}^k)_{k\leq n-1}$ and $(V_n^k)_{k\leq n}$ are the functions we need to introduce at step $n$ to to separate $\theta$ form the microscopic variables at the time of the building of $U_n$. These functions are also convenient to obtain semi-explicit formulas for $[u_n]$ and $\left[\dfrac{\partial u_{n-1}}{\partial r}\right]$.

The introduction of these two families of functions is motivated by two observations:

- First, in the near fields equations, the source terms come from the matching conditions: a behaviour for large $\mathcal{V}$ is imposed. For any $n \in \mathbb{N}$, there are only 4 coefficients to match: $l_0(U_n)^\pm$ and $l_1(U_n)^\pm$ (or $C_{n,0}^\pm$ and $C_{n,1}^\pm$). That is why, for any $n \in N$, the new terms which can appear are combinations of the two following functions:

  - $V_0^0 = 1 \Leftrightarrow \begin{cases} R_0^2 \nabla\cdot(\mu\nabla V_0^0) = 0 \quad \text{in } B_0 \\ V_0^0 = A_0(V_0^0) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} \geq \dfrac{1}{2} \end{cases}$

  - $W_0^0$ such that $W_0^0 - \chi(\mathcal{V})\mathcal{V} - \chi(-\mathcal{V})\mathcal{V} \in W^1(\mathbb{R}^2)$ and

  $$\begin{cases} R_0^2 \nabla\cdot(\mu\nabla W_0^0) = 0 \quad \text{in } B_0, \\ W_0^0 = \pm A_0(W_0^0) + \mathcal{V} + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} \geq \dfrac{1}{2}, \end{cases}$$

  where $\chi$ is a truncation function which satisfies (41).

- The terms which appear in the step $n$ are 'propagated' by the embedded equations (19): for instance, if $U_0 = a_0(\theta)V(S,\mathcal{V})$, then,

  $$\mathcal{A}_0(U_1) = -\mathcal{A}_1(U_0) = -a_0(\theta)\mathcal{A}_1^0(V(S,\mathcal{V})) - \frac{\partial a_0(\theta)}{\partial\theta}\mathcal{A}_1^\theta(V(S,\mathcal{V})).$$

  Therefore $U_1$ linearly depends on $a_0(\theta)$ and $\dfrac{\partial a_0(\theta)}{\partial\theta}$, and it is possible to separate $\theta$ from the fast variables.

**Definition B.8.** *Let $(V_n^k)_{n\in\mathbb{N},0\leq k\leq n}$ be the following family of functions defined by induction:*

$V_0^0 = 1,$

$V_n^k$ *such that* $V_n^k - \chi(\mathcal{V})P^+(V_n^k) - \chi(-\mathcal{V})P^-(V_n^k) \in W_1(\mathbb{R}^2)$ *and*

$$\begin{cases} \nabla\cdot(\mu\nabla V_n^k) = f_n^k \quad \text{in } B_0 \\ V_n^k = A_0^\pm(V_n^k) + P^\pm(V_n^k) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} > \dfrac{1}{2}, \end{cases} \tag{130}$$

*where,*

$$P^\pm(V_n^k) := \sum_{j=1}^{n} A_j^\pm(V_n^k)\frac{\mathcal{V}^j}{j},$$

*and*

$$A_0^+(V_n^k) = -A_0^-(V_n^k),$$

$$A_1^+(V_n^k) = -A_1^-(V_n^k) = \frac{1}{2}\left( < f_n^k, \frac{1}{\mu_\infty} > - \lim_{\mathcal{V}_0 \to +\infty} \sum_{j=2}^{n} \frac{\mathcal{V}_0^{j-1}}{(j-1)!} A_j^+(V_n^k) + \sum_{j=2}^{n} \frac{(-\mathcal{V}_0)^{j-1}}{(j-1)!} A_j^-(V_n^k) \right) \; \forall \mathcal{V}_0 > \frac{1}{2}, (*)$$

$$f_n^k := -\frac{1}{R_0^2}\left( (\mathcal{A}_1^0 V_{n-1}^k + \mathcal{A}_1^\theta V_{n-1}^{k-1}) + (\mathcal{A}_2^0 V_{n-2}^k + \mathcal{A}_2^\theta V_{n-2}^{k-2}) + (\mathcal{A}_3 V_{n-3}^{k-3} + \mathcal{A}_4 V_{n-4}^k) \right).$$

**Remark B.9.** *:*

- $A_l^\pm(V_n^k)$ *are given by*

$$\begin{aligned}\frac{\mathcal{V}^l}{l!} A_l^\pm(V_n^k) = -(\mathcal{A}_0^f)^{-1} \Bigg( & \frac{\mathcal{V}}{(l-1)!} \mathcal{A}_1^f(\mathcal{V}^{l-1}) A_{l-1}(V_{n-1}^k) + \frac{\mathcal{V}^2}{(l-2)!} \mathcal{A}_{2,0}^{0,l}(\mathcal{V}^{l-2}) A_{l-2}(V_{n-2}^k) \\ & + \frac{\mathcal{V}^2}{(l-2)!} \mathcal{A}_{2,0}^{0,\theta\theta}(\mathcal{V}^{l-2}) A_{l-2}(V_{n-2}^{k-2}) + \frac{\mathcal{V}^3}{(l-3)!} \mathcal{A}_3^f(\mathcal{V}^{l-3}) \\ & + \frac{\mathcal{V}^4}{(l-4)!} \mathcal{A}_4^f(\mathcal{V}^{l-4}) A_{l-4}(V_{n-4}^k) \Bigg).\end{aligned}$$

- *The condition* $A_0^+(V_n^k) = -A_0^-(V_n^k)$ *allows us to have the uniqueness of the solution.*
- *The constants* $A_0^+(V_n^k)$ *is unknown.*
- *The condition* ($*$) *about* $A_1^\pm(V_n^k)$ *is a compatibility condition.*

**Proposition B.10.** $\forall n \in \mathbb{N}, 0 \leq k \leq n$*, Problem (130) is well posed.*

*Proof.* The proof is done by induction on $n$. Since $V_0^0$ is given, the initialization is immediate.
Suppose that for any $q < n$ and $p < n$ $V_q^p$ is uniquely defined. We will prove that for any $p \leq n$, $V_n^p$ exists and is unique. The proof falls naturally into two parts. We first prove the uniqueness and then the existence.

1. Uniqueness: let us suppose that $V_n^k$ and $\tilde{V}_n^k$ are two different solutions of (130). We consider $D_n^k = V_n^k - \tilde{V}_n^k$: $D_n^k$ is solution of the following homogeneous problem: find $D_n^k \in W_1(\mathbb{R}^2)$,

$$\begin{cases} \nabla \cdot (\mu \nabla D_n^k) = 0 \in \mathcal{D}'(\mathbb{R}^2), \\ D_n^k = \pm C + o(\mathcal{V}^{-\infty}) \quad \text{when } \pm \mathcal{V} \geq \dfrac{1}{2}. \end{cases}$$

   $D_n^k$ is solution of an homogeneous problem. Since the compatibility condition holds, Proposition 2.4 applies: $D_n^k$ is a constant. It follows that $C = 0$ and $D_n^k = 0$.

2. Existence: let us define $\chi^+$ and $\chi^-$ two smooth truncation functions such that:

   - $\chi^+(\mathcal{V}) = \begin{cases} 1 \text{ if } \mathcal{V} \geq 1 \\ 0 \text{ if } \mathcal{V} \leq \frac{1}{2} \end{cases}$
   - $\chi^-(\mathcal{V}) = \chi^+(-\mathcal{V})$

We also introduce $\tilde{V}_n^k = V_n^k - \chi^- P^- - \chi^+ P^+$ where,

$$P^- := \sum_{j=1}^{n} A_j^-(V_n^k)\frac{\mathcal{V}^j}{j!} \quad \text{and} \quad P^+ := \sum_{j=1}^{n} A_j^+(V_n^k)\frac{\mathcal{V}^j}{j!}.$$

$\tilde{V}_n^k$ is in $W_1(\mathbb{R}^2)$ and

$$\begin{cases} \nabla.(\mu\nabla\tilde{V}_n^k) = \tilde{f}_n^k, \\ \tilde{f}_n^k = f_n^k - \mu_\infty\chi^-\Delta V^- - \mu_\infty\chi^+\Delta V^+ + \phi_c, \\ \phi_c = -2\nabla\chi^-\cdot\nabla V^- - \Delta\chi^- V^- - 2\nabla\chi^+\cdot\nabla V^+ - \Delta\chi^+ V^+. \end{cases}$$

$\phi_c$ comes from of the cut-off: its support is contained is $supp(\chi^+) \cup supp(\chi^-)$.

It is easy to check that $f_n^k - \mu_\infty\chi^-\Delta V^- - \mu_\infty\chi^+\Delta V^+$ is exponentially decreasing. Moreover, combining the properties of $\mathcal{A}_j$ with the fact that $V_n^k \in H^1_{loc}(B_0 \cap \{|\mathcal{V}| < 1\})$, we obtain that $\tilde{f}_n^k$ is in $(W_1(\mathbb{R}^2))^\star$.

To complete the proof it suffices to prove that the compatibility condition $< \tilde{f}_n^k, 1 >= 0$ holds. Indeed, in this case, Proposition (2.4) applies and yields the existence of $V^k_{n,var}$.

$$< \tilde{f}_n^k, 1 > = < f_n^k, 1 > - \lim_{\mathcal{V}_0\to+\infty}\int_{-\mathcal{V}_0}^{\mathcal{V}^0}\int_0^1 \left(\nabla\cdot\mu\nabla(\chi^+V^+) + \nabla\cdot\mu\nabla(\chi^-V^-)\right).$$

The compatibility condition $(*)$ gives the desired conclusion:

$$\begin{aligned} &\lim_{\mathcal{V}_0\to+\infty}\int_{-\mathcal{V}_0}^{\mathcal{V}^0}\int_0^1 (\nabla\cdot\mu\nabla(\chi^+V^+ + \chi^-V^-)) \\ &\qquad = \mu_\infty\left(2A_1^+(V_n^k) + \int_0^1\sum_{j=2}^{n}(A_j^+(V_n^k)\frac{(\mathcal{V}_0)^{j-1}}{(j-1)!} - A_j^-(V_n^k)\frac{(-\mathcal{V}_0)^{j-1}}{(j-1)!})\right), \\ &\qquad =< f_n^k, 1 > . \end{aligned}$$

Consequently $\tilde{V}_n^k$ exists and is unique. By uniqueness, $V_n^k$ exists and is unique.

□

We now introduce a second family of functions $(W_n^k)$.

**Definition B.11.** *: let $(W_n^k)_{n\in\mathbb{N},0\le k\le n}$ be the following family of functions defined by induction: $W_0^0 \in W_1(\mathbb{R}^2)$ such that*

$$\begin{cases} R_0^2\nabla\cdot(\mu\nabla W_0^0) = 0 \quad \textit{in } B_0, \\ W_0^0 = \pm A_0(W_0^0) + \mathcal{V} + o(\mathcal{V}^{-\infty}) \quad \textit{when} \quad \pm\mathcal{V} \ge \dfrac{1}{2}, \end{cases} \tag{131}$$

*$W_n^k$ such that $W_n^k - \chi(\mathcal{V})P^+(W_n^k) - \chi(-\mathcal{V})P^-(W_n^k) \in W_1(\mathbb{R}^2)$ and*

$$\begin{cases} \nabla\cdot(\mu\nabla W_n^k) = g_n^k \quad \textit{in } B_0 \\ W_n^k = A_0^\pm(W_n^k) + P^\pm(W_n^k) + o(\mathcal{V}^{-\infty}) \quad \textit{when} \quad \pm\mathcal{V} \ge \dfrac{1}{2}, \end{cases} \tag{132}$$

*where,*

$$P^{\pm}(W_n^k) := \sum_{j=0}^{n+1} A_j^{\pm}(W_n^k)\frac{\mathcal{V}^j}{j},$$

*and*

$$A_0^+(W_n^k) = -A_0^-(W_n^k),$$

$$A_1^+(W_n^k) = -A_1^-(W_n^k) = \frac{1}{2}\left( < g_n^k, \frac{1}{\mu_\infty} > - \lim_{\mathcal{V}_0 \to +\infty} \sum_{j=2}^{n} \frac{\mathcal{V}_0^{j-1}}{(j-1)!} A_j^+(W_n^k) + \sum_{j=2}^{n} \frac{(-\mathcal{V}_0)^{j-1}}{(j-1)!} A_j^-(W_n^k) \right) \ \forall\, \mathcal{V}_0 > \frac{1}{2}(*),$$

$$g_n^k = -\frac{1}{R_0^2}\left(\mathcal{A}_1^0 W_{n-1}^k + \mathcal{A}_{1^\theta} W_{n-1}^{k-1}\right) + \left(\mathcal{A}_0 W_{n-2}^k + \mathcal{A}_2^\theta W_{n-2}^{k+2}\right) + \left(\mathcal{A}_3 W_{n-3}^{k-3} + \mathcal{A}_4 W_{n-k}^k\right)\right).$$

In the same manner than in Proposition B.10, we can also prove the existence and the uniqueness of this family of functions.

**Proposition B.12.** *$\forall n \in \mathbb{N}, \forall p \leq n$, Problem (132) is well-posed.*

## B.3 Existence and Uniqueness

We can now generalize the approach used in Subsections 2.2.2 and 2.2.3 to each $n$ and prove the proposition 2.8.

**Proposition 2.8.** The system of equations made of (17),(19) and (24) has unique solutions $(u_n, U_n)$ such that $u_n \in H^1(\Omega^+) \cup H^1(\Omega^-)$, $U_n(\cdot,\cdot,\theta) \in H^1_{loc}(\mathbb{R}^2)$ and is non-exponentially increasing with respect to $\mathcal{V}$. Moreover,

$$\begin{cases} \triangle u_n + \dfrac{\omega^2 \rho_\infty}{\mu_\infty} u_n = \delta_n^0 \dfrac{f}{\mu_\infty} \quad \text{in } \Omega^+ \cup \Omega^-, \\ [u_n] = \displaystyle\sum_{j=1}^{n}\sum_{k=0}^{j} 2\langle \frac{\partial^k u_{n-j}}{\partial \theta^k}\rangle A_0^+(V_j^k) + \sum_{j=0}^{n-1}\sum_{k=0}^{j} 2\langle \frac{\partial^{k+1} u_{n-1-j}}{\partial\theta^k \partial r}\rangle A_0^+(W_j^k) \quad (a), \\ \left[\dfrac{\partial u_n}{\partial r}\right] = \displaystyle\sum_{j=2}^{n+1}\sum_{k=0}^{j} 2\langle \frac{\partial^k u_{n+1-j}}{\partial\theta^k}\rangle A_1^+(V_j^k) + \sum_{j=1}^{n}\sum_{k=0}^{j} 2\langle \frac{\partial^{k+1} u_{n-j}}{\partial\theta^k\partial r}\rangle A_1^+(W_j^k) \quad (b), \end{cases} \quad (55) \quad (133)$$

and

$$U_n(\mathcal{V}, S, \theta) = \sum_{j=0}^{n}\sum_{k=0}^{j} \langle \frac{\partial^k u_{n-j}(R_0,\theta)}{\partial\theta^k}\rangle V_j^k + \sum_{j=0}^{n-1}\sum_{k=0}^{j}\langle \frac{\partial^{k+1} u_{n-1-j}(R_0,\theta)}{\partial\theta^k\partial r}\rangle W_j^k. \quad (56) \qquad (134)$$

*Proof.* : the proof is done by induction on $n$.The hypothesis of induction is:

$$(\mathcal{H}_n): \left| \begin{array}{l} \forall i \geq 2,\ U_{n-i} \text{ exists and is unique,} \\ \forall i \geq 2,\ u_{n-i} \text{ exists and is unique. it verifies (55) ,} \\ \text{if } u_{n-1} \text{ exists, } U_{n-1} \text{ exists, is unique and is defined by (56),} \\ \text{if } u_{n-1} \text{ exists, } [u_{n-1}] \text{ is given by (55-(a)) .} \end{array} \right.$$

1. The initialization of the induction have been done in the subsections 2.2.2 and 2.2.3: we have proved the three following propositions:

    - $U_0$ and $u_0$ exist and are unique.

- if $u_1$ exists, $U_1$ exists and is unique.
- if $u_1$ exists, $[u_1]$ verifies (55-(a)) .

2. Induction: we assume that $(\mathcal{H}_n)$ is true. We shall construct $U_n$ solution to the following P.D.E

$$\nabla \cdot (\mu \nabla U_n) = -\frac{1}{R_0^2}(\mathcal{A}_1 U_{n-1} + \mathcal{A}_2 U_{n-2} + \mathcal{A}_3 U_{n-3} + \mathcal{A}_4 U_{n-4}), \tag{135}$$

and which satisfies the following asymptotic behaviour

$$U_n = u_n^{\pm}(R_0,\theta) + \nu \frac{\partial u_{n-1}^{\pm}}{\partial r} + \sum_{k=2}^{n} \mathcal{V}^k \frac{1}{k!}\frac{\partial^k u_{n-k}^{\pm}}{\partial r^k} + o(\mathcal{V}^{-\infty}) \quad \text{for } \pm \mathcal{V} > \frac{1}{2}. \tag{136}$$

As for $U_0$ and $U_1$ it is natural to construct $U_n$ as

$$U_n = \alpha V_0^0 + \beta W_0^0 + \sum_{j=1}^{n}\sum_{k=0}^{j} \langle \frac{\partial^k u_{n-j}(R_0,\theta)}{\partial \theta^k} \rangle V_j^k + \sum_{j=1}^{n-1}\sum_{k=0}^{j} \langle \frac{\partial^{k+1} u_{n-1-j}(R_0,\theta)}{\partial \theta^k \partial r} \rangle W_j^k,$$

where $\alpha$ and $\beta$ are two functions of $\theta$ that have to be determined. By construction, $U_n$ satisfies 135. To compute $\alpha$ and $\beta$ we have to identify the terms of order $0$ and $1$ of the polynomial expansion of $U_n$ with 136 (as explained in B.1, the terms of degree larger than two in the polynomial behaviour of $U_n$ automatically matches).

- The identification of the polynomial term of degree 1 gives

$$\beta = \langle \frac{\partial u_{n-1}}{\partial r} \rangle,$$

and

$$\left[\frac{\partial u_{n-1}}{\partial r}\right] = \sum_{j=2}^{n}\sum_{k=0}^{j} 2\langle \frac{\partial^k u_{n-j}}{\partial \theta^k} \rangle A_1^+(V_j^k) + \sum_{j=1}^{n-1}\sum_{k=0}^{j} 2\langle \frac{\partial^{k+1} u_{n-1-j}}{\partial \theta^k \partial r} \rangle A_1^+(W_j^k).$$

- In the same way, the identification of the polynomial term of degree 1 gives

$$\alpha = \langle u_n \rangle,$$

and

$$[u_n] = \sum_{j=1}^{n}\sum_{k=0}^{j} 2\langle \frac{\partial^k u_{n-j}}{\partial \theta^k} \rangle A_0^+(V_j^k) + \sum_{j=0}^{n-1}\sum_{k=0}^{j} 2\langle \frac{\partial^{k+1} u_{n-1-j}}{\partial \theta^k \partial r} \rangle A_0^+(W_j^k).$$

So, if $U_n$ exists, $[u_n]$ and $\left[\frac{\partial u_{n-1}}{\partial r}\right]$ are given by (55 (a)(b)). Since $u_{n-1}$ verifies the well-posed problem (55), $u_{n-1}$ exists. This implies the two following propositions and $\mathcal{H}_{n+1}$ is proved:

if $u_n$ exists, $U_n$ exists, is unique and is defined by (56).

if $u_n$ exists, $[u_n]$ is given by (55-(a)) .

□

# C Approximate conditions in the general case

In this part, we do not assume that $\rho$ and $\mu$ are symmetric. The jump conditions of the far fields $u_0$ and $u_1$ are given by

$$[u_0] = 0 \qquad \left[\frac{\partial u_0}{\partial r}\right] = 0$$

$$[u_1] = A_0\langle r\frac{\partial u_0}{\partial r}\rangle + A_1\langle\frac{\partial u_0}{\partial \theta}\rangle \qquad \left[\frac{\partial u_1}{\partial r}\right] = B_0\langle u_0\rangle + B_2\langle\frac{\partial^2 u_0}{\partial \theta^2}\rangle + B_1\langle r\frac{\partial^2 u_0}{\partial r\partial \theta}\rangle$$

where

$$A_1 = \frac{2A_0(V_1^1)}{R_0} \qquad B_1 = -\frac{1}{(R_0)^2}\int_{]-\frac{1}{2},\frac{1}{2}[^2}\mu\frac{\partial W_0^0}{\partial S}$$

Applying the reciprocity principle, we can see that

$$B_1 = A_1$$

Actually, since $V_1^1$ and $W_0^0$ are real,

$$0 \;=\; \int_{]-\frac{1}{2},\frac{1}{2}[^2}\nabla\cdot(\mu\nabla W_0^0)V_1^1 \;=\; 2A_0(V_1^1) - \int_{]-\frac{1}{2},\frac{1}{2}[^2}\mu\nabla W_0^0\cdot\nabla V_1^1$$

Therefore,

$$2A_0(V_1^1) = \int_{]-\frac{1}{2},\frac{1}{2}[^2}\mu\nabla W_0^0\cdot\nabla V_1^1$$

Moreover,

$$\begin{aligned} B_1 &= \frac{1}{R_0}\int_{]-\frac{1}{2},\frac{1}{2}[^2}\nabla\cdot(\mu\nabla V_1^1)W_0^0 \\ &= \frac{1}{R_0}\int_{]-\frac{1}{2},\frac{1}{2}[^2}\mu\nabla W_0^0\cdot\nabla V_1^1 \\ &= \frac{2A_0(V_1^1)}{R_0} \\ &= A_1 \end{aligned}$$

Consequently, the 'natural' transmission conditions for the first order approximate conditions are

$$\begin{aligned} [v_1^\delta] \;&=\; \delta\left(A_0\langle r\frac{\partial v_1^\delta}{\partial r}\rangle \;+\; A_1\langle\frac{\partial v_1^\delta}{\partial \theta}\rangle\right) \\ \left[r\frac{\partial v_1^\delta}{\partial r}\right] \;&=\; \delta\left(B_0\langle v_1^\delta\rangle \;+\; B_2\langle\frac{\partial^2 v_1^\delta}{\partial \theta^2}\rangle \;+\; A_1\langle r\frac{\partial^2 v_1^\delta}{\partial r\partial \theta}\rangle\right) \end{aligned}$$

As in the symmetric case, these conditions associated with the classical Helmholtz equation do not define a well-posed problem. So, in the same manner as in the symmetric case, we shift the jump and we use uncentered approximate conditions.

We propose the following first order approximate problem: find $v_1^\delta \in V_{\alpha\delta}$,

$$\begin{cases} \triangle v_1^\delta + \omega^2 \dfrac{\rho_\infty}{\mu_\infty} v_1^\delta = \dfrac{f}{\mu_\infty} \quad \text{in } \Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^- \\ [v_1^\delta]_\alpha = \delta A_0^\alpha (r\dfrac{\partial v_1^\delta}{\partial r})_\alpha^+ + \delta A_1 (\dfrac{\partial v_1^\delta}{\partial \theta})_\alpha^- \\ \left[ r\dfrac{\partial v_1^\delta}{\partial r}\right]_\alpha = \delta \left( B_0^\alpha (v_1^\delta)_\alpha^- + (B_2^\alpha - \dfrac{(A_1)^2}{A_0^\alpha})(\dfrac{\partial^2 v_1^\delta}{\partial \theta^2})_\alpha^- \right) + \dfrac{A_1}{A_0^\alpha}\left[\dfrac{\partial v_1^\delta}{\partial \theta}\right]_\alpha \\ \dfrac{\partial v_1^\delta}{\partial r} + i\omega v_1^\delta = 0 \quad \text{on } S_{R_e} \end{cases} \tag{137}$$

**Remark C.1.** *:*

– *Replacing* $\langle \dfrac{\partial v_1^\delta}{\partial \theta} \rangle$ *by* $(\dfrac{\partial v_1^\delta}{\partial \theta})_\alpha^-$ *allows us to work in the same variational framework as in the symmetric case.*

– *The normal derivative jump is obtained replacing* $\langle r\dfrac{\partial^2 v_1^\delta}{\partial r \partial \theta}\rangle$ *by* $\dfrac{1}{\delta A_0^\alpha}[\dfrac{\partial v_1^\delta}{\partial \theta}] - (\dfrac{\partial^2 v_1^\delta}{\partial \theta^2})_\alpha^- \dfrac{A_1}{A_0^\alpha}$

The variational formulation associated to (137) is given by: find $v_1^\delta \in V_{\alpha\delta}$

$$a_G^\delta(v_1^\delta, v) = -\int_{\Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-} \frac{f}{\mu_\infty}\bar{v} \quad \forall v \in V_{\alpha\delta}$$

where

$$\begin{aligned} a_G^\delta(u,v) \;=\; & \int_{\Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-} \left( \nabla u \cdot \nabla \bar{v} - \frac{\omega^2 \rho_\infty}{\mu_\infty} u\bar{v}\right) \;+\; i\omega \int_{s_{R_e}} u\bar{v} \\ & + \frac{1}{A_0^\alpha \delta}\int_0^{2\pi} [u][\bar{v}] \;+\; \delta B_0^\alpha \int_0^{2\pi} (u)_\alpha^- (\bar{v})_\alpha^- - \delta (B_2^\alpha - \frac{(A_1)^2}{A_0^\alpha}) \int_0^{2\pi} \left(\frac{\partial u}{\partial \theta}\right)_\alpha^- \left(\frac{\partial \bar{v}}{\partial \theta}\right)_\alpha^- \\ & - \frac{A_1}{A_0^\alpha}\int_0^{2\pi} \left( (\frac{\partial v_1^\delta}{\partial \theta})_\alpha^- [\bar{v}] + [v_1^\delta](\frac{\partial \bar{v}}{\partial \theta})_\alpha^- \right) \end{aligned}$$

**Proposition C.2.** *:*

- *For* $\alpha$ *large enough such that* $B_2^\alpha < 0$ *and* $A_0^\alpha > 0$*, the problem* (137) *is well-posed. Moreover there exists a constant* $C_\alpha$ *independent of* $\delta$ *such that:*

$$\forall u \in V_{\alpha\delta} \quad \|u\|_{V_{\alpha\delta}} \leq C_\alpha \sup_{v \in V_{\alpha\delta}, v \neq 0} \frac{a_G^\delta(u,v)}{\|v\|_{V_{\alpha\delta}}} \tag{138}$$

- *For any* $\gamma > 0$*, there exists* $\delta_0$ *such that, for* $\delta < \delta_0$

$$\| u^\delta - v_1^\delta \|_{H^1(\Omega_\gamma)} \leq C\delta^2 \tag{139}$$

*Proof.* :

- The proof starts with the observation that $A_G^\delta$ can be split into a coercive sesquilinear form and a compact sesquilinear form:

$$a_G^\delta(u,u) = \int_{\Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-} \left( |\nabla u|^2 - \frac{\omega^2 \rho_\infty}{\mu_\infty} |u|^2 \right) \; + \; i\omega \int_{s_{Re}} |u|^2$$
$$+ \frac{1}{A_0^\alpha \delta} \int_0^{2\pi} |[u]|^2 \; + \; \delta B_0^\alpha \int_0^{2\pi} |(u)_\alpha^-|^2 \; - \; \delta(B_2^\alpha - \frac{(A_1)^2}{A_0^\alpha}) \int_0^{2\pi} | \left(\frac{\partial u}{\partial \theta}\right)_\alpha^- |^2$$
$$- \frac{2A_1}{A_0^\alpha} \Re \left( \int_0^{2\pi} (\frac{\partial u}{\partial \theta})_\alpha^- [\bar{u}] \right)$$

But, $\forall \varepsilon > 0$,

$$\frac{2A_1}{A_0^\alpha} \Re \left( \int_0^{2\pi} (\frac{\partial u}{\partial \theta})_\alpha^- [\bar{u}] \right) \geq -\frac{2|A_1|}{A_0^\alpha} \left( \frac{1}{\varepsilon \delta} \int_0^{2\pi} |[u]|^2 + \varepsilon \delta \int_0^{2\pi} |(\frac{\partial u}{\partial \theta})_\alpha^- |^2 \right)$$

Let us choose a particular $\varepsilon$ such that $(B_2^\alpha - \frac{(A_1)^2}{A_0^\alpha} + \frac{2|A_1|\varepsilon}{A_0^\alpha})$ is negative and let us introduce the compact bilinear form $\mathcal{C}_\varepsilon$:

$$\mathcal{C}_\varepsilon(u,v) = \frac{1}{\delta}(\frac{1}{A_0^\alpha} - \frac{2|A_1|}{A_0^\alpha \varepsilon}) \int_0^{2\pi} [u][\bar{v}] - \frac{2\omega^2}{\mu_\infty} \int_{\Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-} u\bar{v} - i\omega \int_{s_{Re}} u\bar{v}$$
$$- \delta(B_0^\alpha - |B_0^\alpha|) \int_0^{2\pi} (u)_\alpha^- (\bar{v})_\alpha^-$$

It is clear that $a_G^\delta(u,v) - \mathcal{C}_\varepsilon(u,v)$ is coercive:

$$a_G^\delta(u,u) - \mathcal{C}_\varepsilon(u,u) \;\; \geq \;\; \int_{\Omega_{\alpha\delta}^+ \cup \Omega_{\alpha\delta}^-} \left( |\nabla u|^2 + \frac{\omega^2 \rho_\infty}{\mu_\infty} |u|^2 \right) - \delta(B_2^\alpha - \frac{A_1^2}{A_0^\alpha} + \frac{2|A_1|\varepsilon}{A_0^\alpha}) \int_0^{2\pi} | \left(\frac{\partial u}{\partial \theta}\right)_\alpha^- |^2$$
$$+ |B_0^\alpha| \int_0^{2\pi} |(u)_\alpha^-|^2$$

Therefore (137) satisfies the Fredholm Alternative. We are reduced to proving the uniqueness. In the same manner as in the symmetric case, if $f = 0$, $v_1^\delta = 0$ in $\Omega_\alpha^+$. Finally, the problem does not have no-trivial solutions since for any $n \in \mathbb{Z}$ the determinant of the following matrix cannot be zero:

$$\begin{pmatrix} 1 + inA_1\delta & 0 \\ \frac{\delta}{2}(B_0^\alpha - n^2(B_2^\alpha - \frac{{A_1}^2}{A_0^\alpha})) - in\frac{A_1}{A_0^\alpha} & 1 \end{pmatrix}$$

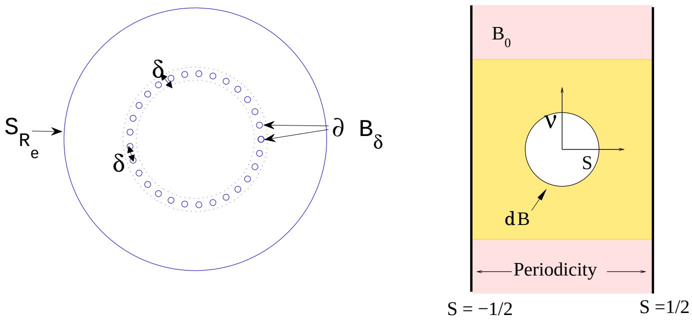
- The proofs of stability and convergence are the same as in the symmetric case.

□

# D Dirichlet Case

It is interesting to study a problem close to (1-2) but where the periodic heterogeneities are replaced by periodic holes: more precisely, we study the following problem:

$$\begin{cases} \nabla \cdot (\mu^\delta \nabla u^\delta) + \omega^2 \rho^\delta u^\delta = f \ \text{ in } \Omega^\delta, \\ u = 0 \ \text{ on } \partial B^\delta \\ \dfrac{\partial u^\delta}{\partial r} + i\omega u^\delta = 0 \text{ on } S_{R_e} \end{cases} \tag{140}$$




(a) $\Omega^\delta$ (b) $B_0$

Figure 17: Definition domain and periodic cell

We still assume that the support of $f$ does not intersect the ring and that $\mu^\delta$ and $\rho^\delta$ are periodic and verify (4) and (6). $\partial B^\delta = \partial B(S, \mathcal{V})_{S=R_0 \frac{\theta}{\delta}, \mathcal{V}=\frac{r-R_0}{\delta}}$, where $\partial B$ is a closed curve contained in the set $]\frac{1}{2}, \frac{1}{2}[\times \mathbb{R}$ (see fig.(17(a)) and fig.(17(b)))

As for the problem (1-2), we want to build approximate conditions. The same method of analysis applies. We first do an asymptotic expansion and we deduce from it an approximate model.

## D.1 Asymptotic expansion

We start from the classical ansatz:

$$u^\delta(R, \theta) = \begin{cases} \displaystyle\sum_{n \in \mathbb{N}} \delta^n u_n^\pm \quad \text{far from the periodic ring,} \\ \displaystyle\sum_{n \in \mathbb{N}} \delta^n \mathcal{U}_n^\pm \quad \text{in the vicinity of the periodic ring.} \end{cases}$$

The far and near fields $u_n$ and $\mathcal{U}_n$ verify the following equations:

$$\Big\{ \mu_\infty \Delta u_n^\pm + \omega^2 \rho_\infty u_n^\pm = \delta_n(0) f \quad \text{in } \Omega^\pm \qquad \begin{cases} \nabla \cdot (\mu \nabla (\mathcal{U}_n)) = -\dfrac{1}{R_0^2} \displaystyle\sum_{k=1}^{4} \mathcal{A}_k U_{n-k} \text{ in } B_0 \\ \mathcal{U}_n = 0 \quad \text{on } \partial B \end{cases}$$

where $\mathcal{A}_k$ are defined by (19).
In addition, the matching conditions do not change, they are still given by (24) ( or 129).

We need a suitable framework to prove the existence and the uniqueness of near fields. This is the object of the following proposition whose proof is based on a Hardy inequality (see [18] (chapter 2) details).

**Proposition D.1.**
*Let* $\mathbb{V} = \left\{ \mathcal{V} \in \mathcal{D}(B_0), V\ 1 - S - periodic, \mathcal{V}_{|\partial B} = 0, \nabla \mathcal{V} \in L^2(B_0),\ \frac{\mathcal{V}}{\sqrt{1+\mathcal{V}^2}} \in L^2(B_0) \right\}$ *and its associated norm:*

$$\|v\|_{\mathbb{V}}^2 = \|\nabla v\|^2_{L^2(B_0)} + \|\frac{v}{\sqrt{1+\mathcal{V}^2}}\|^2_{L^2(B_0)}$$

- $\mathbb{V}$ *is an Hilbert and the semi-norm* $\|\nabla v\|^2_{L^2(B_0)}$ *is a norm in* $\mathbb{V}$.
- *If* $\sqrt{1+\mathcal{V}^2} f \in L^2(B_0)$, *the problem*

$$\begin{cases} \text{look for } u \in \mathbb{V} \\ \displaystyle\int_{B_0} \mu \nabla u \cdot \nabla \bar{\mathcal{V}} = \int_{B_0} F \bar{\mathcal{V}} \quad \forall\, \mathcal{V} \in \mathbb{V} \end{cases} \tag{141}$$

*is well-posed. Moreover if* $f$ *is compact supported, there are two constants* $C^+$ *and* $C^-$ *such that*

$$\lim_{\mathcal{V} \to \pm\infty} u = C^\pm$$

Applying the previous theorem, we can assert that for any $n \in \mathbb{N}$, $(u_n^\pm, u_n)$ is well-defined:

**Proposition D.2.**

- *For any* $n \in \mathbb{N}$, $(u_n^\pm, u_n)$ *exists and is unique. It is given by:*

$$u_0 = 0, \qquad \begin{cases} \Delta u_0 + \dfrac{\omega^2 \rho_\infty}{\mu_\infty} u_0 = f \\ u_0^+ = u_0^- = 0 \quad \text{on } S_{R_0} \\ \frac{\partial u_0}{\partial r} + i\omega u_0 = 0 \quad \text{on } S_{R_e} \end{cases} \tag{142}$$

$$u_n = \sum_{j=0}^{n-1} \sum_{k=0}^{j} \left( \langle \frac{\partial^{k+1} u_{n-j-1}}{\partial r \partial \theta^k} \rangle \mathcal{V}_j^k + \left[ \frac{\partial^{k+1} u_{n-j-1}}{\partial r \partial \theta^k} \right] w_j^k \right) \tag{143}$$

$$\begin{cases} \Delta u_n + \dfrac{\omega^2 \rho_\infty}{\mu_\infty} u_n = 0 \\ (u_n)^\pm = \displaystyle\sum_{j=0}^{n-1} \sum_{k=0}^{j} \left( \langle \frac{\partial^{k+1} u_{n-j-1}}{\partial r \partial \theta^k} \rangle A^\pm_{\mathcal{V}_j^k} + \left[ \frac{\partial^{k+1} u_{n-j-1}}{\partial r \partial \theta^k} \right] A^\pm_{w_j^k} \right) \quad \text{on } S_{R_0} \\ \frac{\partial u_n}{\partial r} + i\omega u_n = 0 \quad \text{on } S_{R_e} \end{cases} \tag{144}$$

*where* $\mathcal{V}_j^k$ *and* $w_j^k$ *are functions that depend only on the fast variables* $\mathcal{V}$ *and* $S$. *They are the solutions of the following well-posed cell problems:*

$$\begin{cases} \nabla \cdot (\mu \nabla \mathcal{V}_0^0) = 0, \quad in B_0 \\ \mathcal{V}_0^0 = 0 \quad \text{on } \partial B, \\ \mathcal{V}_0^0 \sim A^+_{\mathcal{V}_0^0} + \mathcal{V} \quad \text{for large } \mathcal{V}, \mathcal{V} > 0, \\ \mathcal{V}_0^0 \sim A^-_{\mathcal{V}_0^0} + \mathcal{V} \quad \text{for large } \mathcal{V}, \mathcal{V} < 0. \end{cases}$$

$$\forall j \geq 1, \textit{and } 0 \leq k \leq j, \tag{145}$$

$$\begin{cases} \nabla \cdot (\mu \nabla \mathcal{v}_j^k) = f_{\mathcal{v}_j^k}, & \textit{in} B_0 \\ \mathcal{v}_j^k = 0 \quad \textit{on } \partial B, \\ \mathcal{v}_j^k \sim A^+_{\mathcal{v}_j^k} + \displaystyle\sum_{q=2}^{j+1} C^+_{\mathcal{v}_j^k,q} \frac{\mathcal{V}^q}{q!} & \textit{for large } \mathcal{V},\ \mathcal{V} > 0, \\ \mathcal{v}_j^k \sim A^-_{\mathcal{v}_j^k} + \displaystyle\sum_{q=2}^{j+1} C^-_{\mathcal{v}_j^k,q} \frac{\mathcal{V}^q}{q!} & \textit{for large } \mathcal{V},\ \mathcal{V} < 0, \\ f_{\mathcal{v}_j^k} = -\frac{1}{R_0^2} \left( (\mathcal{A}_1^0 \mathcal{v}_{j-1}^k + \mathcal{A}_1^\theta \mathcal{v}_{j-1}^{k-1}) + (\mathcal{A}_2^0 \mathcal{v}_{j-2}^k + \mathcal{A}_2^\theta \mathcal{v}_{j-2}^{k-2}) + (\mathcal{A}_3 V_{j-3}^{k-3} + \mathcal{A}_4 \mathcal{v}_{j-4}^k) \right). \end{cases} \tag{146}$$

$$\begin{cases} \nabla \cdot (\mu \nabla \mathcal{w}_0^0) = 0, & \textit{in} B_0 \\ \mathcal{w}_0^0 = 0 \quad \textit{on } \partial B, \\ W_0^0 \sim A^+_{\mathcal{w}_0^0} + \dfrac{\mathcal{V}}{2} & \textit{for large } \mathcal{V}, \mathcal{V} > 0, \\ W_0^0 \sim A^-_{\mathcal{w}_0^0} - \dfrac{\mathcal{V}}{2} & \textit{for large } \mathcal{V}, \mathcal{V} < 0. \end{cases} \tag{147}$$

$\forall j \geq 1$, *and* $0 \leq k \leq j$,

$$\begin{cases} \nabla \cdot (\mu \nabla \mathcal{w}_j^k) = f_{V_j^k}, & \textit{in} B_0 \\ \mathcal{w}_j^k = 0 \quad \textit{on } \partial B, \\ \mathcal{w}_j^k \sim A^+_{\mathcal{w}_j^k} + \displaystyle\sum_{q=2}^{j+1} C^+_{\mathcal{w}_j^k,q} \frac{\mathcal{V}^q}{q!} & \textit{for large } \mathcal{V},\ \mathcal{V} > 0, \\ \mathcal{w}_j^k \sim A^-_{\mathcal{w}_j^k} + \displaystyle\sum_{q=2}^{j+1} C^-_{\mathcal{w}_j^k,q} \frac{\mathcal{V}^q}{q!} & \textit{for large } \mathcal{V},\ \mathcal{V} < 0, \\ f_{\mathcal{w}_j^k} = -\dfrac{1}{R_0^2} \left( (\mathcal{A}_1^0 \mathcal{w}_{j-1}^k + \mathcal{A}_1^\theta \mathcal{w}_{j-1}^{k-1}) + (\mathcal{A}_2^0 \mathcal{w}_{j-2}^k + \mathcal{A}_2^\theta \mathcal{w}_{j-2}^{k-2}) + (\mathcal{A}_3 V_{j-3}^{k-3} + \mathcal{A}_4 \mathcal{w}_{j-4}^k) \right). \end{cases} \tag{148}$$

- *Moreover, we have the following error estimate: for any $n \geq 0$, for any $\gamma > 0$, there are a constant $C$ independent of $\delta$ and a constant $\delta_0 > 0$ such that for any $\delta < 0$,*

$$\|u^\delta - \sum_{l=0}^{n} \delta^l u_l^- \|_{H^1(\Omega_\gamma^-)} + \|u^\delta - \sum_{l=0}^{n} \delta^l u_l^+ \|_{H^1(\Omega_\gamma^+)} \leq C\delta^{n+1} \tag{149}$$

**Remark D.3.**

- *An important point to note here is that the limit problem $u_0$ is not the the problem without periodic ring, but the problem with homogeneous Dirichlet condition on $S_{R_0}$. More generally, the problems defining $u_n^+$ and $u_n^-$ are uncoupled. However, $u_n^+$ and $u_n^-$ depend on both $u_k^+$ and $u_k^-$ for $k \leq n-1$.*
- *In the problems* (146) *and* (148)*, the constants $C^\pm_{\mathcal{v}_j^k,q}$ and $C^\pm_{\mathcal{w}_j^k,q}$ are known. They are determined by $f_{\mathcal{v}_j^k}$ and $f_{\mathcal{w}_j^k}$.*
- *The proof runs as in the section 2.*

## D.2 Approximate Condition

In this section we build a first order approximate condition in the symmetric case, using the previous asymptotique extension. Combining (142), (144) and (143) yields

$$\begin{cases} [u_0] = \langle u_0 \rangle = 0 \\ [u_1] = A_0 \langle r \dfrac{\partial u_0}{\partial r} \rangle + A_1 \left[ r \dfrac{\partial u_0}{\partial r} \right] \\ \langle u_1 \rangle = B_0 \left[ r \dfrac{\partial u_0}{\partial r} \right] + B_1 \langle r \dfrac{\partial u_0}{\partial r} \rangle \end{cases}$$

where

$$\begin{array}{ll} A_0 = \dfrac{1}{R_0}(A^+_{\mathcal{V}^0_0} - A^-_{\mathcal{V}^0_0}) & B_0 = \frac{1}{2R_0}(A^+_{\mathcal{W}^0_0} + A^+_{\mathcal{W}^0_0}) \\ A_1 = \dfrac{1}{R_0}(A^+_{\mathcal{W}^0_0} - A^-_{\mathcal{W}^0_0}) & B_1 = \frac{1}{2R_0}(A^+_{\mathcal{V}^0_0} - A^+_{\mathcal{V}^0_0}) \end{array}$$

By the reciprocity principle, it easily to check that $B_1 = A_1$. Moreover, since $\mu$ the periodic cell is symmetric ($\mu$ and $\partial B$ are symmetric in $S$), $A_1 = B_1 = 0$.
We now assume that $B_0 A_0 \neq 0$. We propose this first order approximate problem

$$\begin{cases} \text{look for } \tilde{v}_1^\delta \in H^1(\Omega^+) \cap H^1(\Omega^-) \text{ such that} \\ \Delta \tilde{v}_1^\delta + \dfrac{\omega^2 \rho^\infty}{\mu_\infty} \tilde{v}_1^\delta = f \quad \text{in} \Omega^+ \cup \Omega^- \\ \langle r \dfrac{\partial \tilde{v}_1^\delta}{\partial r} \rangle = \dfrac{1}{\delta A_0} \left[ \tilde{v}_1^\delta \right] \\ \left[ r \dfrac{\partial \tilde{v}_1^\delta}{\partial r} \right] = \dfrac{1}{\delta B_0} \langle \tilde{v}_1^\delta \rangle \\ \frac{\partial \tilde{v}_1^\delta}{\partial r} + i\omega \tilde{v}_1^\delta = 0 \quad \text{on } S_{R_e} \end{cases} \tag{150}$$

and its associated variationnal formulation:

$$\begin{cases} \text{look for } \tilde{v}_1^\delta \in H^1(\Omega^+) \cap H^1(\Omega^-) \text{ such that} \\ a^\delta(\tilde{v}_1^\delta, \tilde{v}) = \dfrac{-1}{\mu_\infty} \displaystyle\int_{\Omega^+ \cup \Omega^-} f \bar{\tilde{v}} \quad \forall\, \tilde{v} \in H^1(\Omega^+) \cap H^1(\Omega^-) \end{cases}$$

where:

$$\tilde{a}^\delta(u, \tilde{v}) = \int_{\Omega^+ \cup \Omega^-} \left( \nabla \tilde{u} \cdot \nabla \bar{\tilde{v}} - \frac{\omega^2 \rho_\infty}{\mu_\infty} \tilde{u} \bar{\tilde{v}} \right) + i\omega \int_{S_{R_e}} \tilde{u} \bar{\tilde{v}} + \frac{1}{\delta A_0} \int_0^{2\pi} [u][\bar{\tilde{v}}] + \frac{1}{\delta B_0} \int_0^{2\pi} \langle \tilde{u} \rangle \langle \bar{\tilde{v}} \rangle$$

It is easily seen that for any $\delta > 0$ the previous problem (150) is well-posed. Nevertheless, it is not clear that it is possible to prove a stability estimate similar to (65), with a constant $C_\omega$ independant of $\delta$.
Again, similarly to to Section 3.1.1 we shift the mean and jump terms of $\alpha\delta$ ($\alpha > 0$) using Taylor expansion. We obtain the following problem:

$$\begin{cases} \text{look for } v_1^\delta \in H^1(\Omega^+_{\alpha\delta}) \cap H^1(\Omega^-_{\alpha\delta}) \text{ such that} \\ \Delta v_1^\delta + \dfrac{\omega^2 \rho^\infty}{\mu_\infty} v_1^\delta = f \quad \text{in} \Omega^+ \cup \Omega^- \\ \langle r \dfrac{\partial v_1^\delta}{\partial r} \rangle = \dfrac{1}{\delta A_0^\alpha} \left[ v_1^\delta \right] \\ \left[ r \dfrac{\partial v_1^\delta}{\partial r} \right] = \dfrac{1}{\delta B_0^\beta} \langle v_1^\delta \rangle \\ \frac{\partial v_1^\delta}{\partial r} + i\omega v_1^\delta = 0 \quad \text{on } S_{R_e} \end{cases} \tag{151}$$

where :

$A_0^\alpha = A_0 + 2\alpha$

$B_0^\alpha = B_0 + 2\alpha$

and it associated variationnal formulation:

$$\begin{cases} \text{look for } v_1^\delta \in H^1(\Omega^+) \cap H^1(\Omega^-) \text{ such that} \\ a^\delta(v_1^\delta, v) = \dfrac{-1}{\mu_\infty} \displaystyle\int_{\Omega^+ \cup \Omega^-} f\bar{v} \quad \forall\, v \in H^1(\Omega^+) \cap H^1(\Omega^-) \end{cases}$$

where:

$$a^\delta(u,v) = \int_{\Omega^+ \cup \Omega^-} \left( \nabla u \cdot \nabla \bar{v} - \frac{\omega^2 \rho_\infty}{\mu_\infty} u\,\bar{v} \right) + i\omega \int_{S_{R_e}} u\,\bar{v} + \frac{1}{\delta A_0} \int_0^{2\pi} [u]_\alpha [\bar{v}]_\alpha + \frac{1}{\delta B_0} \int_0^{2\pi} \langle u \rangle_\alpha \langle \bar{v} \rangle_\alpha$$

We define a convenient norm to prove the stability:

$$\|u\|_{\mathcal{V}_{\alpha\delta}} = \|u\|_{H^1(\Omega^+_{\alpha\delta})} + \|u\|_{H^1(\Omega^-_{\alpha\delta})} + \frac{1}{\delta A_0^\alpha} \int_0^{2\pi} |[u]_\alpha|^2 + \frac{1}{\delta B_0^\alpha} \int_0^{2\pi} |\langle u \rangle_\alpha|^2$$

**Proposition D.4.**

- *For any* $\delta > 0$*, for any* $\omega > 0$*,* (151) *is well-posed: Moreover,*

$$\forall \delta_0 > 0,\ \forall \omega > 0,\ \exists C_\omega^{\delta_0} > 0, \forall \delta < \delta_0, \forall u \in H^1(\Omega^+_{\alpha\delta}) \cap H^1(\Omega^-_{\alpha\delta}),$$
$$\|u\|_{\mathcal{V}-\alpha\delta} \leq C_\omega^{\delta_0} \sup_{v \in H^1(\Omega^+_{\alpha\delta}) \cap H^1(\Omega^-_{\alpha\delta}), v \neq 0} \frac{a(u,v)}{\|v\|_{\mathcal{V}_{\alpha\delta}}} \tag{152}$$

- *For any* $\gamma > 0$*, there exists* $\delta_0$ *such that, for* $\delta < \delta_0$

$$\| u^\delta - u_1^\delta \|_{H^1(\Omega_\gamma)} \leq C\delta^2 \tag{153}$$

# E Properties of $V_n^k$ and $W_n^k$ functions for $n \leq 2$

## E.1 $V_n^k$ family

• $\mathbf{n = 1}$

- $V_1^0$: it is clear that

$$V_1^0 = 0$$

- $V_1^1$: An easy computation shows that $A_{V_1^1,1} = 0$. The function $V_1^1$ is given by:

$$\begin{cases} \nabla \cdot (\mu \nabla V_1^1) = -\dfrac{1}{R_0}\dfrac{\partial \mu}{\partial S} \\ V_1^1 = \pm A_0(V_1^1) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} \geq \dfrac{1}{2} \end{cases}$$

  – if $\mu$ is an even function in the S variable, $V_1^1$ is an odd function in the $S$ variable and so $A_0(V_1^1) = 0$.

  – if $\mu$ is an even function in the $\mathcal{V}$ variable, $V_1^1$ is an even function in the $\mathcal{V}$ variable, which implies that $A_0(V_1^1) = 0$.

• $\mathbf{n = 2}$

- $V_2^0$: an easy computation shows that

$$A_2(V_2^0) = -\frac{\omega^2 \rho_\infty}{\mu_\infty} \quad A_1^+(V_2^0) = \frac{1}{2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \frac{\omega^2(\rho_\infty - \rho)}{\mu_\infty}$$

  $V_2^0$ is the solution of the following problem:

$$\begin{cases} \nabla \cdot (\mu \nabla V_2^0) = -\omega^2 \rho \\ V_2^0 = \pm A_0(V_2^0) + (\pm\mathcal{V})A_1^+(V_2^0) + A_2(V_2^0)\dfrac{\mathcal{V}^2}{2} + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} \geq \dfrac{1}{2} \end{cases}$$

  – if $\mu$ and $\rho$ are even function in the S variable, $V_2^0$ is an even function in the $S$ variable.

  – if $\mu$ and $\rho$ are even function in the $\mathcal{V}$ variable, $V_2^0$ is an even function in the $\mathcal{V}$ variable, which implies that $A_0(V_2^0) = 0$.

- $V_2^1$: according to the properties of $V_1^1$, we see that

$$A_2(V_2^1) = 0 \quad A_1^{\pm}(V_2^1) = 0$$

  So,

$$\begin{cases} \nabla \cdot (\mu \nabla V_2^1) = -\dfrac{1}{R_0^2}\left(2R_0\mathcal{V}\dfrac{\partial}{\partial \mathcal{V}}(\mu\dfrac{\partial V_1^1}{\partial \mathcal{V}}) + R_0\mu\dfrac{\partial V_1^1}{\partial \mathcal{V}}\right) \\ V_2^1 = \pm A_0(V_2^1) + o(\mathcal{V}^-\infty) \quad \text{when} \quad \pm\mathcal{V} \geq \dfrac{1}{2} \end{cases}$$

  – if $\mu$ is an even function in the S variable, $V_2^1$ is an odd function in the $S$ variable. So $A_0(V_2^1) = 0$.

  – if $\mu$ is an even function in the $\mathcal{V}$ variable, $V_2^1$ is an odd function in the $\mathcal{V}$ variable.

- $V_2^2$: according to the properties of $V_1^1$, we see that

$$A_2^{\pm}(V_2^2) = -\frac{1}{R_0^2} \quad A_1^{\pm}(V_2^2) = \frac{1}{2\mu_\infty}\left(+\frac{\mu_\infty}{R_0} - \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \mu\left(\frac{1}{R_0} + \frac{\partial V_1^1}{\partial S}\right) dS d\mathcal{V}\right)$$

So,

$$\begin{cases} \nabla \cdot (\mu \nabla V_2^2) = -\dfrac{1}{R_0^2}\left(R_0 \dfrac{\partial(\mu V_1^1)}{\partial S} + \mu R_0 \dfrac{\partial V_1^1}{\partial S} + \mu\right) \\ V_2^2 = \pm A_0(V_2^2) + A_1^+(V_2^2)(\pm\mathcal{V}) - \dfrac{\mathcal{V}^2}{2R_0^2} + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} \geq \dfrac{1}{2} \end{cases}$$

  - if $\mu$ is an even function in the S variable, $V_2^2$ is an even function in the $S$ variable.
  - if $\mu$ is an even function in the $\mathcal{V}$ variable, $V_2^2$ is an even function in the $\mathcal{V}$ variable, which implies that $A_0(V_2^2) = 0$.

## E.2 $W_n^k$ family

• $\mathbf{n = 0}$

We remind that $W_0^0$ is the solution of the following problem

$$\begin{cases} R_0^2 \nabla \cdot (\mu \nabla W_0^0) = 0 \\ W_0^0 = A_0(W_0^0) + \mathcal{V} + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \mathcal{V} \geq \dfrac{1}{2} \\ W_0^0 = -A_0(W_0^0) + \mathcal{V} + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \mathcal{V} \leq -\dfrac{1}{2} \end{cases}$$

- if $\mu$ is an even function in the S variable, $W_0^0$ is an even function in the $S$ variable.
- if $\mu$ is an even function in the $\mathcal{V}$ variable, $W_0^0$ is an odd function in the $\mathcal{V}$ variable.

• $\mathbf{n = 1}$

- $W_1^0$ is defined by the following problem

$$\begin{cases} \nabla \cdot (\mu \nabla W_1^0) = -\dfrac{1}{R_0}\left(2\mathcal{V}\dfrac{\partial}{\partial \mathcal{V}}(\mu \dfrac{\partial W_0^0}{\partial \mathcal{V}}) + \mu \dfrac{\partial W_0^0}{\partial \mathcal{V}}\right) \quad \text{in } B_0 \\ W_1^0 = \pm A_0(W_1^0) - \dfrac{1}{R_0}\dfrac{\mathcal{V}^2}{2} + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} \geq \dfrac{1}{2} \end{cases}$$

  - if $\mu$ is an even function in the S variable, $W_0^0$ is an even function in the $S$ variable.
  - if $\mu$ is an even function in the $\mathcal{V}$ variable, $W_0^0$ is an odd function in the $\mathcal{V}$ variable.

- $W_1^1$ is defined by the following problem

$$\begin{cases} \nabla \cdot (\mu \nabla W_1^1) = -\dfrac{1}{R_0}\left(\dfrac{\partial(\mu W_0^0)}{\partial S} + \mu \dfrac{\partial W_0^0}{\partial S}\right) \quad \text{in } B_0 \\ W_1^1 = \pm A_0(W_1^1) + (\pm\mathcal{V}) A_1(W_1^1) + o(\mathcal{V}^{-\infty}) \quad \text{when} \quad \pm\mathcal{V} \geq \frac{1}{2} \end{cases}$$

where

$$A_1(W_1^1) = -\frac{1}{2R_0}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \mu \frac{\partial W_0^0}{\partial S}$$

  - if $\mu$ is an even function in the S variable, $W_1^1$ is an odd function in the $S$ variable.This implies that $A_0(W_1^1) = 0$
  - if $\mu$ is an even function in the $\mathcal{V}$ variable, $W_1^1$ is an even function in the $\mathcal{V}$ variable and so $A_0(W_1^1) = 0$

## E.3 Summary

The following tables summarize the previous remarks:

| | $(\mu,\rho)$ $S-$ even | $(\mu,\rho)$ $\mathcal{V}-$ even |
|---|---|---|
| $V_1^1$ | odd | even |
| $V_2^0$ | even | even |
| $V_2^1$ | odd | odd |
| $V_2^2$ | even | even |
| $W_0^0$ | even | odd |
| $W_1^0$ | even | even |
| $W_1^1$ | odd | odd |

Table 1: Parity of $V_n^k$ and $W_n^k$ according to the parity of ($\mu$,$\rho$)

| | $(\mu,\rho)$ $S-$ even | $(\mu,\rho)$ $\mathcal{V}-$ even |
|---|---|---|
| $A_0(V_1^1)$ | $0$ | $0$ |
| $A_0(V_2^0)$ | $\displaystyle\int_{-1/2}^{1/2} V_2^0(\frac{1}{2},S)dS-\frac{1}{2}A_1^+(V_2^0)-\frac{1}{8}A_2^+(V_2^0)$ | $0$ |
| $A_0(V_2^1)$ | $0$ | $\displaystyle\int_{-1/2}^{1/2} V_2^0(\frac{1}{2},S)dS$ |
| $A_0(V_2^2)$ | $\displaystyle\int_{-1/2}^{1/2} V_2^2(\frac{1}{2},S)dS-\frac{1}{2}A_1^+(V_2^2)-\frac{1}{8}A_2^+(V_2^2)$ | $\displaystyle\int_{-1/2}^{1/2} V_2^2(\frac{1}{2},S)dS-\frac{1}{2}A_1^+(V_2^2)-\frac{1}{8}A_2^+(V_2^2)$ |
| $A_0(W_0^0)$ | $\displaystyle\int_{-1/2}^{1/2} W_0^0(\frac{1}{2},S)dS$ | $\displaystyle\int_{-1/2}^{1/2} W_0^0(\frac{1}{2},S)dS$ |
| $A_0(W_1^0)$ | $\displaystyle\int_{-1/2}^{1/2} W_1^0(\frac{1}{2},S)dS-\frac{1}{8R_0}$ | $\displaystyle\int_{-1/2}^{1/2} W_1^0(\frac{1}{2},S)dS-\frac{1}{8R_0}$ |
| $A_0(W_1^1)$ | $0$ | $0$ |

Table 2: Parity of $A_0(V_n^k)$ and $A_0(V_n^k)$ with respect to the parity of ($\mu$,$\rho$)

| | general case | $(\mu,\rho)$ $S-$ even | $(\mu,\rho)$ $R-$ even |
|---|---|---|---|
| $A_1(V_1^1)$ | $0$ | $0$ | $0$ |
| $A_1(V_2^0)$ | $\frac{1}{2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\frac{\omega^2(\rho_\infty-\rho)}{\mu_\infty}dS$ | $A_1(V_2^0)$ | $A_1(V_2^0)$ |
| $A_1(V_2^1)$ | $0$ | $0$ | $0$ |
| $A_1(V_2^2)$ | $\frac{1}{2\mu_\infty}\left(+\frac{\mu_\infty}{R_0}-\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\mu\left(\frac{1}{R_0}+\frac{\partial V_1^1}{\partial S}\right)dSd\mathcal{V}\right)$ | $A_1(V_2^2)$ | $A_1(V_2^2)$ |
| $A_1(W_0^0)$ | $1$ | $1$ | $1$ |
| $A_1(W_1^0)$ | $0$ | $0$ | $0$ |
| $A_1(W_1^1)$ | $-\frac{1}{2R_0}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\mu\frac{\partial W_0^0}{\partial S}$ | $0$ | $A_1(W_1^1)$ |

Table 3: Parity of $A_1(W_n^k)$ and $A_1(W_n^k)$ with respect to the parity of ($\mu$,$\rho$)

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